Topological Social Choice: Reply to Le Breton and Uriarte

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1. Introduction

In a recent paper Le Breton and Uriarte (1990), hereafter LBU, offer a critique of
Chichilnisky's (1979, 1980, 1982) impossibility theorem. Of the many statements of
Chichilnisky's theorem, they single out Baigent's (1984, 1985) as incorrect. The first
purpose of this note is to reply to this claim and argue that Baigent's formulation is
not only formally correct, but also the most expeditious for its purpose.
Furthermore, it will be argued that the approach advocated by LBU is not justified.
Finally, it will be argued that, contrary to both Chichilnisky and LBU, the
topological framework itself should be replaced by the "old fashioned" finte
framework for the analysis of the underlying issue which Chichilnisky's theorem
attempts to address.

   In what follows, especially concerning the appropriateness of different frame-
works, it will be important to distinguish between a claim and the formal expression
of that claim. Given the former, there may be several alternatives for the latter. The
central claims in this area of social choice theory include the following:

Claim 1. Large changes in the social preference should not result from small changes
in individual preferences.

Claim 2. The social choice procedure should not rule out any individual having any
particular preference.

Claim 3. The social preference should never disagree with every individual's
preference.

Claim 4. The identities of the holders of individual preferences should not affect the
social preference.

Claim 5. A topological framework is the most appropriate for the analysis of issues involving Claim 1.

Claim 6. Given Claim 5, Claims 1, 2, 3, and 4 are inconsistent.

Both Chichilnisky and LBU provide arguments in favor of Claim 1 and, in this note, we will not enter into a discussion of its justification. No particular problems arise from the justification or formalisation of Claim 2, at least for the issues which we wish to discuss here. This claim is, of course, associated with the well known condition of Unrestricted Domain. Claim 3 is easily justified, but may be formalised in different ways. In particular it may take the form of the *Respect for Unanimity* condition used by Chichilnisky or some form of the *Pareto Principle*. In general, these two formalisations of the same claim are different. While we have nothing to say here about the justification of Claim 4, its formalisation does require comment. The usual way of expressing Claim 4 is by requiring invariance to any permutation of individual preferences and this requirement is called *Anonymity*. For many purposes, this will be an adequate formal expression of Claim 4. However, given the way Claim 1 is usually expressed, Anonymity conditions do not adequately express Claim 4. While Anonymity does prevent one way in which the identities of preference holders may affect the social preference, there is another way which is not ruled out by Anonymity. In particular, the way in which Claim 1 is formalised by Chichilnisky, using product topologies, depends on the identities of individuals. This was pointed out in Baigent (1984, 1985) where it was shown how Claim 4 could be adequately expressed. This requires both the use of an Anonymity condition together with the use of a quotient topology rather than a product topology. The main purpose of Baigent (1984, 1985) was to show that Claim 6 is unaffected by adopting an adequate formalisation of Claim 4. It should be emphasised that LBU's criticism of Baigent (1984, 1985) does not relate to the extent to which the main purpose, as just stated, is achieved. As for Claim 5, it is clearly supported by both Chichilnisky and LBU; they may even wish to express it more strongly. However, we will argue that Claim 5 should be rejected. When it comes to Claim 6, this is where Chichilnisky and LBU sharply disagree. LBU's rejection of Claim 6 is based on their use of a particular topology, that of closed convergence. We will provide a stronger justification for this topology than that offered by LBU, but we will argue that even the stronger justification is not adequate. This will lead us from direct consideration of the merits of different topologies to a discussion of the merits of different types of justification for different topologies. It is at this meta level of argument that the shortcomings of LBU's rejection of Claim 6 emerge most clearly.

We conclude this section by offering a summary of the central points of disagreement. Chichilnisky and LBU both make Claim 1. Chichilnisky makes Claim 4 but her formal expression of it is inadequate according to Baigent (1984, 1985) where it is shown however, that her Claim 6 is not affected by correcting this shortcoming. Both Chichilnisky and LBU strongly make Claim 5 which we, equally strongly, reject. While Chichilnisky accepts and LBU reject Claim 6, we reject the type of argument on which both positions seem to depend.

Section 2

We begin with the statement of Chichilnisky's theorem in Baigent (1984, 1985). It is elementary that any function is continuous relative to some topology on its domain (e.g., the discrete topology will always suffice). Therefore, it follows immediately that for some topology on n-tuples of preferences, there exist continuous social welfare functions that satisfy any other consistent set of conditions. This is why Chichilnisky's impossibility theorem could not possibly hold for all topologies and why the statement of continuity in her theorem must include a restriction on the topologies that are admissible. In fact, in Baigent (1985) the statement of Chichilnisky's theorem included such a restriction. "Remembering that continuity is taken with respect to an *appropriate* product topology, Chichilnisky's theorem may be stated as follows" immediately precedes the statement of Chichilnisky's theorem in Baigent (1985).

"Appropriateness" in this statement must be understood in the context of the issue being addressed in Baigent (1985). That issue was, given the topological framework, does the Chichilnisky impossibility theorem hold if Claim 4 is adequately formalised? In providing a positive answer to this question, an appropriate topology on preferences is any topology for which Chichilnisky's impossibility holds. Loosely speaking, Baigent (1984, 1985) showed that, for any topology on preferences for which Chichilnisky's impossibility holds, there is a more adequate expression of Claim 4 for which the impossibility continues to hold. This issue is logically distinct from the issue of whether Chichilnisky's impossibility theorem as she stated it, holds for any "acceptable" topology on preferences. LBU are right to raise this issue because it is important. But they should not confuse the requirement of "acceptability' of a topology for this issue with the "appropriateness" of a topology for a different issue.

We turn now to LBU's approach to the six claims set out in the previous section, starting with Claim 2. LBU share Chichilnisky's concern with not ruling out preferences with critical points. Their response is to seek a topology that is reasonable for spaces of preferences that include ones with critical points. Before dealing with the reasonableness issue, we note that impossibility results are *strengthened* by restricting the space of preferences and possibility results are *weakened* by expanding the space of preferences. Thus, while Claim 2 distinguishes a desirable feature of social welfare functions, that same feature is an undesirable feature of impossibility theorems concerning social welfare functions. Given the other conditions in Chichilnisky's impossibility theorem and any of the topologies she uses, discontinuity of the social welfare function occurs even on the space of linear preferences. The strength of this result cannot possibly be increased by expanding the set of preferences to include non-linear preferences or those with critical points. However, this still leaves the more fundamental issue of whether or not Chichilnisky's impossibility result holds for some reasonable topology on preferences?

The two results of LBU, in which the impossibility in Chichilnisky's theorem becomes a possibility, are obtained by using the topology of closed convergence on preferences. An appeal to its popularity in general equilibrium theory is the justification offered by LBU. However, such an appeal by itself, cannot provide an

adequate justification. After all, vices are popular, as well as virtues. Nevertheless, let us begin with the popularity of the topology of closed convergence in general equilibrium theory.

In fact, the topology of closed convergence is only popular in general equilibrium theory for some purposes. For other purposes, other topologies are more popular. In particular, for equivalence results (Walrasian equilibria and the Core), the topology of closed convergence has been found the most useful while, for the analysis of structural stability and genericity, topologies that explicitly depend on smoothness have been more useful (see Mas-Colell (1985) or Trockel (1984) for discussions of the differentiable approach in general equilibrium theory). This class of topologies includes one of those used by Chichilnisky. This illustrates a far more general point. In any formalisation, the properties with which preferences are endowed should depend on the purpose of the analysis. Likewise, if for some purpose, a particular feature of preferences is important, this should be reflected in the topology with which preferences are endowed.

Clearly then, a simple appeal to the popularity of the topology of closed convergence in general equilibrium theory, will not suffice. The question that must be asked is this. What is the reason for the popularity of the topology of closed convergence in general equilibrium theory and does this also constitute a good reason for using it in social choice theory, given the purpose of the latter? Its popularity in general equilibrium theory arises from the following fact. It is the weakest separated topology such that demand correspondences are upper semi continuous (see Hildenbrand (1974)). In accepting this justification in general equilibrium theory, the underlying intuition is that agents whose choices are similar, from all possible budget sets, must have similar preferences. The crucial point is that the concept of closeness of preferences derives from a concept of closeness of the manifestations of preferences. To the extent that we have clearer intuitions about the closeness of the manifestations of preferences than we have about the closeness of preferences, this type of justification appears attractive. Now, this is the case in general equilibrium theory where the only manifestations of preferences are choices from budget sets. But, if a similar justification is to be found for the topology of closed convergence in social choice theory, what are the manifestations of individual preferences in social choice theory? It is certainly not choices from budget sets. Perhaps budget sets should be replaced by arbitrary compact subsets of social alternatives. In fact, careful scrutiny of Hildenbrand's (1974) argument (especially pp 96–100), reveals that the property in question continues to hold. That is, *the topology of closed convergence is the weakest topology on preferences such that the choice correspondence, induced by preferences in the usual way, is upper hemi continuous on the class of all compact subsets.*

It is by no means clear however, that even this extension of the justification used in general equilibrium theory is satisfactory in social choice theory. The difference is this. Whereas individual choices are the sole manifestation of individual preferences in general equilibrium theory, there are no individual choices in social choice theory. In fact, in social choice theory there is nothing at all that can be identified as a manifestation of an individual's preference. This is especially clear if, following Sen (1977), a social welfare function is interpreted as "judgement aggregation". In this case, an individual's preference ranks social alternatives according to that

individual's views concerning how well off society would be in the various social alternatives. Here, the alternatives are defined in such a way that they cannot be objects of individual choice and the interpretation of individual preferences need not bear any particular relationship to individual choices. Therefore, we conclude that the type of argument that justifies the topology of closed convergence in general equilibrium theory is not as directly applicable in social choice theory.

As for the topologies used by Chichilnisky, it seems even harder to find a justification in general equilibrium theory that may be successfully transferred to social choice theory. Focusing on smooth preferences and topologies that reflect smoothness, has the enormous benefit of providing a basis for differentiable approaches in general equilibrium theory. We can think of no issue in social choice theory that requires smoothness. Therefore, we conclude that the topologies used by Chichilnisky also lack a firm justification for their use in social choice theory. In this, we agree with LBU and take this to be the real contribution in their paper.

This finally brings us to Claim 5 which asserts that the most appropriate framework for the analysis of issues involving Claim 1 is a topological framework. We have already argued that the indirect type of justification that has been offered is not satisfactory. What about a more direct justification? That is, can it be argued that one particular topology is a natural topology for expressing "closeness" of preferences? We think it unlikely that such an argument can be found. The greatest merit of topological analysis is that it permits very general and undemanding ways of expressing continuity. However, for spaces such as preferences, this same generality makes it very difficult to know whether any particular topology does accord with our basic intuitions concerning closeness. If this were not the case, then presumably it would be possible to formulate axioms for a topology on preferences and even state a characterisation theorem. That this has not been done, in an area in which axioms are ubiquitous, strongly suggests to us that a topological framework is not the most appropriate for expressing our intuitions concerning closeness of preferences.

This is not to say that we do not have firm intuitions concerning the closeness of preferences. We do have very firm intuitions. They have been axiomatised; they have been used by Kemeny and Snell (1962) to characterise a metric on preferences; this metric was used by Charles Dodgson (better known as Lewis Carroll) in some of the earliest work in social choice theory; this metric has been used in Baigent (1984) to formalise Claims 1 and 6 and extend Claim 6 to social choice functions, something that cannot be done following Chichilnisky's approach. This metric takes the distance between two preferences to be the extent to which their rankings disagree. If there are a finite number of alternatives, then the extent of disagreement may be precisely expressed by the cardinality of the symmetric difference between the two preferences. Given that such a natural metric on preferences exists as one of the oldest features of social choice theory in the finite framework, the force of Claim 5 is difficult to see. Lest we appear to be intellectual Luddites, we do think that topological social choice theory may have much to contribute. But not to the analysis of issues involving Claim 1.

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