Halmos, P. 1970. *Naive Set Theory*. New York-Heidelberg-Berlin: Springer-Verlag.
Henrotte, P. 1992. An example of general equilibrium with derivative securities: The binomial case. Working paper, Stanford University.
Kurz, M. 1993. General equilibrium with endogenous uncertainty. Working paper, Stanford University.
Kurz, M. 1991. On the structure and diversity of rational beliefs. Technical Report No. 39, Stanford Institute for Theoretical Economics (November).
Kurz, M. 1974. The Kesten–Stigum model and the treatment of uncertainty in equilibrium theory. In *Essays in Economic Behavior under Uncertainty* (ed. M. C. Balch, D. McFadden, and S. Wu), Amsterdam: North Holland.
Milnor, J. 1965. *Topology from a differentiable viewpoint*. Charlottesville: The University Press of Virginia.

CHAPTER 6

Market equilibrium with endogenous price uncertainty and options

Peter H. Huang and Ho-Mou Wu

1 Introduction

Arrow's (1953) classic two-period general competitive equilibrium model introduced the canonical theoretical setting for the study of market behavior under uncertainty. The types of randomness described by Arrow's formalization are those factors external to human influence, such as hurricanes, earthquakes, droughts, and floods, that affect production capabilities or consumer tastes. In one bold stroke, Arrow (1953) and Debreu (1959), with the introduction of a complete set of contingent commodity markets, reinterpret the static Arrow and Debreu (1954) model of certainty in terms of a sequential model of uncertainty. This reinterpretation allowed their results about existence and Pareto optimality of static competitive equilibria to carry over completely to a dynamic and uncertain world. Arrow's (1953) model also provided what has become the standard role for securities, namely hedging against exogenous risks by shifting income across exogenous states. As Duffie (1991) noted, Arrow (1953) and Arrow and Debreu (1954) provided financial economists with benchmarks for market behavior that had been missing until then.

In Arrow's paradigm, uncertainty means not knowing which of several possible states will prevail. Agents are assumed to know all the conceivable states that can arise. These states are assumed to form a mutually exclusive and exhaustive description of the future. Arrow's conceptualization of states of nature is related to, but differs from Savage's (1954)

It is our pleasure to contribute this essay to honor our mutual advisor and teacher, Ken Arrow, from whose work, generosity, and scholarly example we have learned much. We thank Ken Arrow, Don Brown, Graciela Chichilnisky, Frank Hahn, Geoff Heal, Philippe Henrotte, Mordecai Kurz, Chris Shannon, Jan Werner, three anonymous referees, and the audiences of seminars at Columbia, Duke, and Stanford for helpful discussions.

definition of personal states of the world for subjective probabilities in statistical decision theory because the state space must be agreed on by everybody in Arrow's framework in order to have markets for either contingent commodities or securities. A crucial assumption of the Arrow paradigm is that traders know the mapping from states into equilibrium prices, $p(s)$. Any uncertainty about market prices disappears once the true state is known. In other words, individuals possess perfect foresight about prices, conditional on knowledge of the realized state.[1]

Criticisms of Arrow's (1953) notion of rational expectation as conditional perfect foresight focus on either the restriction that the equilibrium mapping $p(s)$ is a function with a unique p for each s or on the incredible computational abilities economic agents must have to compute the function $p(s)$. In this chapter, we explore the possible extensions that economic theory can now offer to address both these criticisms. First, we consider a departure from Arrow's paradigm which examines the implications when mapping of $p(s)$ is not a function, but a correspondence instead. In other words, people know that several equilibrium prices exist. We review the leading work by Chichilnisky (1992; 1994), Chichilnisky, Dutta, and Heal (1991), and Chichilnisky and Wu (1992). Although it is difficult to ensure uniqueness of equilibria, local finiteness of equilibria is easily ensured. Debreu (1970) proved that a typical (or technically regular) economy has a finite number of equilibria. However, even this mild form of nonuniqueness of equilibria creates difficulties for the usual notion of rationality of expectations in general equilibrium theory. It raises four questions: If the agents know that several equilibrium prices exist, what optimization problems do individuals face when they are uncertain as to which of several possible equilibrium prices will prevail; what is a sensible notion of general equilibrium with price uncertainty; can such an equilibrium be shown to exist; and what is the role of options in such an equilibrium with price uncertainty?

The first part of this chapter examines these questions when agents are assumed to possess, as in Arrow (1953), structural knowledge of the economy. The second part studies the notion of price uncertainty when

[1] Since then, Arrow's position on the rational expectations doctrine has evolved somewhat. Arrow (1978) explained how differential information alters the policy neutrality propositons regarding macroeconomic fluctuations of the "rational expectations" school of thought. Arrow (1982) suggested that experimental evidence from cognitive psychology can explain such observed failures of the rational anticipations hypothesis in financial markets as analogous to Shiller's (1981) findings about excess volatility in bond and stock markets. Arrow (1986) discussed the implications of bounded rationality for formulating a concept of equilibrium.

agents do not possess structural knowledge of the economy and form expectations on this. Kurz (1974; 1996), Kurz and Wu (1996), and others pursue this line of research. It is conceivable that before equilibrium prices are realized, investors may not have enough information to select a correct set of possible equilibrium prices. This means that prior beliefs may have as their support a much larger space than the set of realized equilibrium prices. Ex post market prices are observable and posterior beliefs place probability one on the equilibrium price actually realized. Our notion of rationality of beliefs drops the restriction, due to Hicks (1946), that beliefs have to be time invariant.[2] Such a time dependence of beliefs is natural in light of the multiplicity of equilibria and the irreversible nature of a sequential economy. The diffuse nature of these priors can be due to uncertainty about other households' endowments and utility functions, the lack of time or ability to compute equilibrium prices, complexity reasons, or some unforeseen contingencies.

In Section 2, we discuss further the meaning of price uncertainty and review the related literature about endogenous uncertainty. In Section 3, we discuss the role of options and price-contingent securities in the allocation of endogenous risk. Section 4 describes an economy in which agents face price uncertainty. We suggest that investors can hedge against spot price uncertainty by trading European options when they know the equilibrium price correspondence $p(s)$. We also study the properties of equilibria when there are complete or incomplete options markets. In Section 5, we examine the notion of equilibrium when economic agents do not have enough computational ability to identify the equilibrium correspondence $p(s)$. Section 6 offers concluding thoughts.

2 The meaning of price uncertainty

As Arrow (1991) noted, uncertainty is clearly unavoidable in (economic) life. How to deal with uncertainty for both economic theorists and practitioners is the problem. Before Arrow's (1953) tour de force essay, economists represented uncertainty by probability distributions over exogenous and endogenous randomness. Since Arrow's seminal formulation of uncertainty in terms of exogenously given states of nature, economic theory in general and general equilibrium theory in particular

[2] The idea of decoupling the set of date 1 spot prices that households expect are possible into those they expect at date 0 versus those they expect at date 1 is inspired by Arrow. He observed that what is crucial in his 1953 equivalence result between complete contingent commodity markets and complete security markets is the assumption that households not only have single-valued price expectations at date 0, but also that households maintain those same beliefs at date 1.

have been expressed primarily in terms of such "states of nature."[3] Two facts must be true of these states if markets for state-contingent commodities or assets with payoffs indexed by states to exist. First, states must be agreed on and commonly observable by economic actors. Second, states can only capture uncertainty not influenced by human action. If either property fails to hold for the concept of states, the well-known problems of moral hazard and/or adverse selection can lead to the nonexistence of markets or nonexistence of equilibria when competitive markets do exist.

The comprehensive nature of these states is such that once the true state is known, no residual uncertainty remains. In particular, this means that on learning the actual state s, agents know the values of market prices $p(s)$. In other words, $p(s)$ is assumed to be a common knowledge single-valued function. This informational assumption is restrictive in several ways. First, agents might not be able to solve for the function $p(s)$. This is the approach taken by Kurz (1994a, 1994b, 1996) and Kurz and Wu (1996) in a model of endogenous uncertainty and rational beliefs, where agents do not know the structure of the economy and possess heterogeneous beliefs about the values of endogenous variables in their economy. Second, if agents have structural knowledge of their economy, they might realize that generically $p(s)$ is a correspondence. This is what Chichilnisky, Dutta, and Heal (1991), Chichilnisky (1992, 1994), and Chichilnisky and Wu (1992) assume. The difference between these works concerns the agents' knowledge of their economy. Kurz (1994a, 1994b, 1996) and Kurz and Wu (1996) postulate that agents do not know the structural equations describing their economy and so does Chichilnisky (1992, 1994). Chichilnisky, Dutta, and Heal (1991) assume that agents know the "reduced form" of their economy, namely the correspondence $p(s)$. Finally, agents may not even be aware of all the possible states. This more realistic and less understood setting is that of unforeseen contingencies. Kreps (1992) and Huang (1993) offer the beginnings of such a theory.

[3] A notable exception to Arrow's legacy is game theory which by its very nature is an interactive multiperson decision-making theory. The fundamental problem of game theory (for players and game theorists alike) is to predict how players will behave. Thus players face uncertainty over the endogenous decisions of other players. Incomplete information games are converted into complete, but imperfect information games by introducing the notion of players' types. Thus, strategic uncertainty over endogenously chosen decisions can often be replaced by structural uncertainty over exogenously fixed parameters describing a game. Aumann and Brandenburger (1991) recently introduced the notion of an interactive belief system to deal simultaneously with strategic and structural uncertainty.

The recently growing literature about price uncertainty has its roots in the study of endogenous uncertainty discussed by Kurz (1974). Kurz pointed out that the dominant form of uncertainty economic agents face is not exogenous uncertainty, but a qualitatively different type of uncertainty he termed endogenous uncertainty. The difference between exogenous and endogenous uncertainty is that the former results from events determined by nature whereas the latter is determined by the decisions of economic agents themselves. Kurz made an important observation about the crucial nature of endogenous uncertainty, especially in modern economies. Undoubtedly, the driving source of uncertainty behind macroeconomic and financial time series does not come from natural disasters, important and tragic as they may be. Instead, the volatility in financial markets and of national incomes is primarily due to endogenous uncertainty. In particular, economic agents are uncertain about the values of endogenous economic variables, such as prices, quantities, and qualities. But, as Arrow (1994) pointed out, the concept of endogenous uncertainty has many possible interpretations that require unpacking or unbundling.

Svensson (1981) considered uncertainty about prices in a temporary general equilibrium model involving a complete set of Arrow–Debreu commodity markets contingent on any relative price vector in the unit simplex. He applied the infinite dimensional commodity space model of Bewley (1972) to show the existence of competitive equilibria. Kurz (1994a, 1994b) introduced a theory of rational beliefs which allows for heterogeneous beliefs about prices due to agents not knowing the structural equations of the economy they inhabit. Kurz and Wu (1996) analyzed the equilibrium and welfare properties of competitive equilibrium when trading endogenous uncertainty is accomplished by using price-contingent contracts rather than the Arrow–Debreu state-contingent contracts. They introduce rationality restrictions on beliefs and construct a "price state space" which requires the expansion of the exogenous state space to include the "state of beliefs." Kurz (1996) reviewed the recent work on endogenous uncertainty with rationality restrictions on beliefs.

A different interpretation and formalization of endogenous uncertainty, as in Chichilnisky, Dutta, and Heal (1991), is based on the multiplicity of equilibria, which they assume economic actors have the computational power to calculate. Chichilnisky, Dutta, and Heal (1991) proved that finitely many successive levels of price-contingent Arrow–Debreu futures markets are sufficient to ensure fully against price uncertainty. Hahn (1992, 1994) argued that a set of complete Arrow securities contingent on exogenous states and endogenous prices leads to the nonexistence of a rational expectations equilibrium if there are

multiple spot market equilibrium prices. Chichilnisky (1992) investigated general equilibrium in a model with nested layers of endogenous uncertainty. She drew a parallel between Russell's famous paradox in formal logic and the problem of hedging endogenous uncertainty. Chichilnisky, Heal, and Tsomocos (1994) model endogenous uncertainty over whether options will be exercised. Chichilnisky (1994) considered the theory and policy applications of endogenous uncertainty induced by production activities in an economy. In yet another framework not involving multiplicity of equilibria, Chichilnisky and Wu (1992) analyzed individual and collective risk (as studied by Cas, Chichilnisky, and Wu [1996]) in a general equilibrium model of default. Their model demonstrated that endogenous uncertainty can be due to financial innovation and that it depends crucially on the endogenous pattern of trade.

3 The role of options

Arrow's (1953) model provided what has become the standard role for securities, namely hedging against risks by shifting income across *exogenous* states. He demonstrated that a complete set of (what have since then become known as Arrow) security markets can replace the larger number of complete contingent commodity markets in achieving a Pareto efficient allocation of risk over exogenous uncertainty. Financial economists have since then focused on what is the span, in the space of income across states, of a set of assets. Ross (1976) showed that (sufficiently many) options can be used to complete state-contingent asset markets and provide insurance against exogenous uncertainty. Other state-contingent assets can achieve the same goal of completing state-contingent markets. Ross cited the relative easiness or cost advantages of writing options as the main reason for trading options.

We reexamine here the role that price-contingent securities or options play in insuring against *endogenous* price uncertainty. To complete markets in a world with exogenous uncertainty one can also use other state-contingent assets instead of options. Then options may or may not offer any "comparative advantage" in completing asset markets. This presents us with a puzzle: Why are options traded so actively in real world economies? The answer becomes clear when we consider endogenous uncertainty: because endogenous price uncertainty is so prevalent in real world economies and price-contingent securities, such as options, are uniquely suited to handle those price-related risks. Options therefore play the same important role for handling endogenous price uncertainty that state-contingent assets do for handling exogenous state uncertainty in Arrow (1953). This role of options cannot be replaced by any other kinds of assets which are not price-contingent.

In the case of complete state-contingent markets for exogenous uncertainty, options are "redundant" in terms of their payoffs and can be priced by arbitrage (see Hakansson (1978)). In spite of this, we show that options also serve the indispensable role of protecting investors from disequilibrium expectations about endogenous price uncertainty. The fact that option payoffs can be replicated by portfolios of existing primary assets does not mean that option markets have no role to play in the optimal allocation of risk concerning price uncertainty. Just the opposite is true. Our viewpoint turns out to be analogous to that of Mas-Colell (1992) regarding asset redundancy and sunspots.

In our model, as discussed in the next section, option markets play two additional roles. First, a set of option markets changes not only the number of, but also the actual set of, equilibrium spot prices and commodity allocations. Thus, the financial sector (consisting of commodity option markets) in our economy fundamentally affects the real sector. Second, complete option markets select a particular spot market equilibrium as being focal. Option premia facilitate the coordination of households' beliefs on a particular vector of commodity prices by the no-arbitrage condition on equilibrium option premia. Notice that we do not claim option premia are unique. Different equilibrium option premia correspond to different spot prices via the no-arbitrage condition on equilibrium option premia.

4 Price uncertainty with structural knowledge

It is helpful to view this section as investigating the logical implications of two axioms:

A1 An economy is regular (this ensures finiteness of equilibria).
A2 Investors have knowledge of the mapping $p(s)$, whether it be a function or a correspondence.

We also evaluate two properties of equilibria:

P1 Multiple price equilibria exist.
P2 Price-contingent option markets efficiently allocate risk about endogenous uncertainty.

In the following section, we describe an economy satisfying A1, A2, and P1 and we introduce price-contingent option markets. We then investigate the compatibility of P1 with A1 and A2 when there are complete

option markets. The question of what role price-contingent securities play in the allocation of endogenous risk is what P2 addresses. This question is analogous to Arrow's (1953) concern over the role of securities in the optimal allocation of exogenous risk. In the case of complete option markets, P2 is true. In fact, complete option markets completely eliminate any price uncertainty in the spot market. In the case of incomplete option markets, obviously a first-best allocation of risk from price uncertainty is too much to ask for. So P2 fails; but it can be shown that P1 is generically true with incomplete option markets.

4.1 *The economy with price uncertainty*

Consider a pure exchange economy with G goods, two dates 0 and 1, and N exogenous states of nature at date 1. There are complete asset markets with respect to these exogenous states. In what follows, we concentrate on the study of price uncertainty. To simplify our notation, we set $N = 1$ for the rest of the chapter. There are G spot commodity markets at date 1, while no consumption occurs at date 0. It might be helpful to think of date 1 as when this economy actually takes place, whereas date 0 is an instant just before then, as in Arrow (1953). The raison d'être for date 0 is to allow households to trade in assets written on spot commodity prices at date 1. There are H households, each with a (column) vector of initial commodity endowments, e^h (ε) R^G. Let $e = (e^1, \ldots, e^H) \in R^{GH}$. We denote the consumption of commodities by household h as a (column) vector X^h ε R^G. We denote the (row) vector of spot commodity prices by $p \in S = \{p \in R_{++}{}^G \mid \Sigma_{j=1}^G p_j = 1\}$.

We assume that at date 0, households possess common probability beliefs over a common (finite) set of possible date 1 equilibrium spot market prices, $C = \{p^1, \ldots, p^n\}$, that is a subset of S. If C is a nonsingleton set, then households do not have single-valued expectations about equilibrium prices. Let $\pi = (\pi(p^1), \ldots, \pi(p^n))$ denote the shared vector of date 0 subjective probability beliefs over C. At date 1, households also possess common probability beliefs over a common set of possible date 1 equilibrium spot market prices, D, which is a subset of S.

We require as part of the definition of equilibrium that all households' beliefs at both dates be correct in the sense that C is a (possibly, proper) subset of the actual set of equilibrium date 1 spot prices that are possible from the perspective of date 0 and that D is another (possibly, proper and possibly, different) subset of the actual set of equilibrium date 1 spot prices that are possible from the perspective of date 1. We write $B = (C, D)$. Notice that C and D are endogenous in two senses: They are

chosen endogenously by households and endogenously constrained in equilibrium to be correct in the sense that the support of shared beliefs cannot include any spot prices known not to be possible equilibria (at the date of those beliefs). In many interesting cases, D is a subset of C. Notice also that nothing rules out the possibility that C is not equal to D, but clearly, unless there is a reason for changing or updating beliefs, C will equal D. In that case, we let $I = C = D$ denote the time-invariant set of (rationally) expected possible equilibrium date 1 prices.

Households have strictly monotone and strictly convex preferences representable by differentiable utility functions, u^h, satisfying the assumption of Debreu's (1970) regular economy approach. For expositional purposes, we first consider the absence of any options trading at date 0. In such a no-asset economy, households formulate date 0 plans regarding date 1 consumption.

Definition 1: *A consumption plan for household h,* $x^h(p)_{p\in C} = (x^h(p^1), \ldots, x^h(p^n))$ *is feasible if it satisfies a set of n budget constraints:*

$$px^h(p) = pe^h \text{ for each } p \in C \tag{4.1}$$

Note that equation (1) represents multiple budget constraints, one for each member of C; only a single budget constraint realizes from the set of conceivable ones because only a single p in C realizes. Household h with (e^h, u^h) chooses a consumption plan to maximize expected utility $EU^h = \Sigma_{i=1}^n \pi^h(p^i)u^h(x^h(p^i))$ among feasible consumption plans. Without assets or any other decisions to be made at date 0, expected utility reduces to utility. Nonetheless, households must choose a plan that consists of consumption vectors x^h for each member of C that might realize.

Definition 2: *A consumption allocation is given by* $x(p)_{p\in C} = (x^1(p)_{p\in C}, \ldots, x^H(p)_{p\in C})$. *A consumption allocation is feasible if all its component consumption plans* $x^h(p)_{p\in C}$ *are feasible.*

Definition 3: *An SME (spot market equilibrium)* $(I, (p)_{p\in I}, x(p)_{p\in I})$ *is a time-invariant set (commonly held across households) of expected equilibrium spot prices I, commodity spot market prices* $p \in I$, *and an allocation* $x(p)_{p\in I}$ *such that for all h,* $x^h(p)_{p\in I}$ *maximizes* $EU^h = \Sigma_{i=1}^n \pi(p^i)u^h(x^h(p^i))$ *subject to budget constraints* (1) *and*

$$\sum_h x^h(p) = \sum_h e^h \text{ for all } p \in I \tag{4.2}$$
$$(\text{commodity market clearing})$$

We reiterate that this is a no-asset economy and households are unable to shift across different possible equilibrium spot prices the income realized from the sale of price-independent endowments. Although there are n budget constraints, one for each price vector in I, at date 1 only a single p is realized and only one of the n budget constraints actually obtains. A complete set of price-contingent markets would mean that all n budget constraints have to be satisfied at date 0 with no spot markets necessary at date 1. The latter scenario involves nG markets at date 0, while an SME involves only G markets at date 1. An immediate question that arises is whether an equilibrium can be shown to exist. We can answer this question in the affirmative in the case of regular economies.

Proposition 1: *For a regular economy, an SME exists.*

Proof: Notice that our no-asset economy differs from a static pure exchange Arrow–Debreu (1954) economy without uncertainty in only two ways: our economy takes place over two dates and involves the maximization of expected utility instead of utility. Because households do not have to make consumption decisions until date 1 occurs, however, expected utility reduces to certain utility. Thus, we have an Arrow–Debreu economy, for which existence of an equilibrium is well known (see, for example, Debreu [1959]). Because our economy is regular, we know (see, for example, Debreu [1970]) that there is a finite number of equilibria. Thus, I is finite. QED.

Next, we suppose that households can trade options to hedge against price uncertainty. Since there are only two dates, American options are equivalent to European options, which expire on the second date. The asset structure of this economy consists of European call and put options, which pay off in a numeraire commodity (such as gold). The payoff amounts are assumed to depend on the date 1 spot price of the first good. Our model can be generalized to include options written on commodity futures contracts or an index of spot prices. The available European commodity options are characterized by their strike prices, k, that lie in a set P. Each k is measured in prices normalized by $\Sigma_{j=1}^{G} p_j = 1$. Thus, P can be used to represent the asset structure and is a subset of [0,1]. For now, we

assume that P is countable and is given exogenously. Option payoffs are defined as follows:

Definition 4: *The European option payoff $V_E(p,k) = Max\ [0,p_1\text{-}k]$ if $E = c$ for a call option and $V_E(p,k) = Max\ [0,k\text{-}p_1]$ if $E = p$ for a put option.*

We represent option premia by $q_E(k)$, $E = c, p$, and the vector of option premia by q. We denote a household's option portfolio by $\theta_E(k)_{k\in P} = (\theta_E^1(k)_{k\in P}, \ldots, \theta_E^H(k)_{k\in P})$ with the vector of option holding by $\theta_E^h(k)_{k\in P}$.

Definition 5: *A general equilibrium with price uncertainty (GEPU) is a set (commonly held across households) of expected equilibrium spot prices $B = (C,D)$, a price system $(q,(p)_{p\in C},(p)_{p\in D})$, and an allocation $(\theta(k)_{k\in p},x(p)_{p\in C},x(p)_{p\in D})$ such that, for all h, $(\theta^h(k)_{k\in p}, x^h(p)_{p\in C})$ maximizes $EU^h(x^h(p)_{p\in C})$ subject to:*

$$\sum_{k\in P}\sum_E q_E(k)\theta_E^h(k) = 0 \tag{4.3}$$
$$(\text{option budget constraint})$$

$$px^h(p) = \sum_{k\in P}\sum_E V_E(p,k)\theta_E^h(k) + pe^h \tag{4.4}$$
$$\text{for each } p \in C,\ (\text{ex ante budget constraint})$$

and

$$\sum_h \theta_E^h(k) = 0 \text{ for } E = c, p \text{ and all } k \in P, \tag{4.5}$$
$$(\text{option market clearing})$$

$$\sum_h x^h(p) = \sum_h e^h \text{ for all } p \in C \tag{4.6}$$
$$(\text{ex ante commodity market clearing})$$

Furthermore, $x^h(p)$ maximizes $u^h(x^h(p))$ subject to:

$$px^h(p) = \sum_{k\in P}\sum_E V_E(p,k)\theta_E^h(k) + pe^h \tag{4.7}$$
$$\text{for each } p \in D,\ (\text{ex post budget constraint})$$

and

$$\sum_h x^h(p) = \sum_h e^h \text{ for all } p \in D \tag{4.8}$$
$$(\text{ex post commodity market clearing})$$

As before, an immediate question arises: How do we know whether a GEPU exists? We can answer this question in the affirmative when $C = D = \mathrm{I}$ is a singleton. Notice that this does not necessarily require there be a unique equilibrium, only that economic agents somehow coordinate on one of the multiple equilibria, if there are several.

Proposition 2: *For a regular economy, if $C = D = I$ is a singleton, then a GEPU exists.*

Proof: Notice that if C is a singleton, there will be no activity in options markets because there is no price uncertainty. But the zero volume of trade in options means this economy is a regular no-asset economy, for which the existence of an SME (x^*,p^*) has already been proved in Proposition 1. That SME with $\theta_E^h(k) = 0$ for all h and k, $q = 0$, and $C = D = \{p^*\}$ is a GEPU (with the obvious modification that a free option can be in excess demand). QED.

The question remains: What about existence when C or D is not a singleton? We answer this question by separately considering the case of complete and incomplete option markets.

4.2 *Price uncertainty with complete option markets*

By complete option markets, we mean that all options, characterized by $k \in P$, are available for trading. For example, P can be a dense set in [0,1]. Complete option markets result in an ex ante Pareto efficient consumption allocation. We can thus show that an equilibrium allocation exists and must be invariant across which member of C realizes. This is due to the strict convexity of preferences and because the economy-wide endowment is independent of spot prices. Such price independence of equilibrium consumption plans means that price uncertainty is no longer payoff-relevant for households. Thus, in equilibrium, households are fully insured against price uncertainty and can achieve an ex ante Pareto optimum allocation. But any point on the expected utility possibility frontier maximizes a weighted sum of households' expected utilities. This characterization of the equilibrium allocation as the solution to a concave maximization problem over a compact set guarantees the existence of an equilibrium allocation, x^*. This method of proof, due to Negishi (1960), is used later to show the existence of GEPU. For an equilibrium allocation, the corresponding unique shadow price vector forms an equilibrium spot market commodity price vector, p^*. Thus, an active set of complete option markets has allowed households to coordinate on a particular spot market price vector. In our notation, $(C,D) = (E^*,p^*)$, where E^* is the finite set of equilibria of this regular economy without full insurance.

Proposition 3: *With complete option markets, the GEPU allocation x^* is invariant with respect to which element of C realizes.*

Proof: This result is the consequence of the strict convexity of preferences and endowments being independent of price. If a GEPU allocation x^* varies with $p \in C$, then by strict convexity of preferences, household h strictly prefers the allocation $Ex^{*h} = \Sigma_{p\in C}\, \pi(p)x^{*h}(p)$, formed by the expected value of x^* over $p \in C$, to the allocation, x^{*h}. That is to say, for all h, $u^h(Ex^{*h}) > Eu^h(x^{*h})$ if $x^{*h}(p)$ varies over $p \in C$. Because π is a probability vector over $P \varepsilon C$, $\Sigma_{p\in C}\pi(p) = 1$, and $\Sigma_h Ex^{*h} = \Sigma_h\Sigma_{p\in C}\, \pi(p)x^{*h}(p) = \Sigma_h\, x^{*h}(p) = \Sigma_h\, e^h$. We have thus constructed a feasible allocation, Ex^*, that is Pareto superior to x^*, which contradicts the assumption that x^* is a GEPU with complete options. Thus, any GEPU allocation x^* must be independent of which $p \in C$ realizes because households trade options to smooth out consumption independently of which price from C realizes. QED.

This result implies that a GEPU allocation, x^* reaches full insurance against price uncertainty. Any GEPU allocation lies on the expected utility possibility frontier, or maximizes a social welfare function that is a weighted sum of households' expected utilities.

At this point, it might seem that Proposition 2 should apply. But, it does not for two reasons. First, $C \neq D$. Second, there is no reason that option premia at date 0 are unique. They will typically not be. Different option premia and portfolio trades correspond to different (x^*,p^*) by no-arbitrage and the above construction. In a sense, complete option markets have allowed households to shift the multiplicity from date 1 equilibrium spot prices to date 0 equilibrium option premia. This might seem to be merely an intertemporal transfer of equilibrium price multiplicity, but such a reallocation of multiplicity over time has real consequences because there is a genuine difference between uncertainty about which equilibrium spot market price will prevail at date 1 and uncer-

tainty about which one of several equilibria option premia q could have been realized at date 0. In terms of the formal model, this is because there is no date (-1) before date 0.

In this model we do not assume that households worry about forecasting at some prior date which q would occur at date 0. To embark on this scenario is to become quickly embroiled in an infinite regress as considered in Chichilnisky, Dutta, and Heal (1991). Although that approach is logically satisfying, options on options are uncommon and higher than second-order options are even less common. Alternatively, Hahn's (1992) point is that complete option markets eliminate their own raison d'être by eliminating price uncertainty so effectively. But if option markets are not utilized because traders believe there will be no price uncertainty, then price uncertainty will in fact occur. This reasoning leads to the view that a rational expectations equilibrium cannot exist. In order to resolve this apparent paradox, we note that arguing that options are not required once they have already been traded is analogous to saying that after buying full insurance, there is no further demand for any insurance. Our approach separates the set C from the set D and stops going back in time from date 1 to date 0 after trading options at date 0.

A complementary viewpoint is that we assume households are sophisticated enough to understand that if nobody trades in options there will be multiplicity in spot prices. The usual argument is that any individual household's option portfolio is insignificant (of measure zero if there were a continuum of households) and so everybody will try to free-ride on the options trading of others to eliminate spot price multiplicity. We hold the view that households also realize the possibility that spot price multiplicity will occur if there is no (or not enough) activity in the options market.[4] They trade in options not out of any sense of public service, but because they want to guard against the off-the-equilibrium possibility that not enough other households trade in options to eliminate spot price uncertainty. Formally, at date 0, households believe C will prevail; but at date 1, households believe D will prevail. Households possess rational expectations at both dates. A complete set of option markets changes not only the cardinality of, but also the actual set of, equilibria for our economy. Next we outline the proof that there exists a general equilibrium with price uncertainty.

[4] This phenomenon is analogous to the trembling-hand reasoning in game theory. Players can revise beliefs about strategies in an extensive form game on reaching a node that has zero probability given their prior beliefs. However, in our framework, D is required to be a subset of C.

Proposition 4: *A GEPU exists, consisting of a fully insured unique allocation x^* and a spot price vector, p^*, supporting the full insurance allocation.*

Proof: Following Negishi's (1960) approach to proving the existence of an Arrow–Debreu competitive equilibrium, we note that any fully insured allocation must lie on the frontier of the expected utility set. An ex ante Pareto efficient allocation maximizes, for some weights, λ^h, a weighted sum of expected utilities, $\Sigma^h \lambda^h E u^h(x^h(p)_{p\in C})$, over the set of feasible expected utility allocations. Such an optimization problem involving the maximization of a concave objective function over a compact set is known to have a solution, x^*. By the second fundamental theorem of welfare economics, which applies to SME of this regular convex economy, there exists a unique spot price vector, p^*, that supports x^*. In other words, (p^*,x^*) is an Arrow–Debreu competitive equilibrium for the date 1 economy. With asset prices and portfolios (q,θ) derived by the no-arbitrage condition, it forms a GEPU. QED.

We have now provided some answers to the four questions posed in Section 1. With complete options, market equilibrium exists and is efficient (P2 is true), but the spot equilibrium price is unique (P1 fails). We observe that in the unique complete market GEPU, options look like "redundant" assets. This redundancy is consistent with such existing financial theories as the Black–Scholes option pricing model. But options are required to guard against off-equilibrium expectations. In this sense, option markets substitute for the coordination of expectations. Options are uniquely required for accomplishing such a task and therefore cannot be replaced by other nonprice-contingent assets. This finding is stronger than Ross's (1976) result because in our model options are the only assets capable of providing insurance against multiple price equilibria, whereas in Ross's model options are sufficient, but certainly not necessary, to span asset markets.

4.3 *Price uncertainty with incomplete option markets*

By incomplete option markets, we mean that the number of possible equilibrium spot prices exceeds the number of available options in P. Recall that so far, P has been exogenously specified. However, if P remains exogenously specified, then a GEPU can fail to exist and, even when it exists, may not be efficient.

With exogenous uncertainty and incomplete options markets, it is well known that market equilibria are generically constrained suboptimal (see Geanakoplos and Polemarchakis [1986]). A similar reasoning can be applied to the case of endogenous price uncertainty to establish the suboptimality of GEPU if it exists. In other words, a central planner restricted to utilizing existing incomplete options markets can, in principle, construct a reallocation of commodities and assets that is a Pareto improvement over the market equilibrium. Hence P2 fails and the equilibrium allocation does not entail full insurance against price uncertainty. In addition, we cannot use the method of Propositon 4 to establish the existence of GEPU with incomplete options markets.

In the context of exogenous uncertainty, a robust counterexample to generic existence of an equilibrium was constructed by Polemarchakis and Ku (1990). Generic existence fails to hold because a neighborhood exists in the parameter space of initial endowments and the strike prices for which the European call or put option pays out zero in all states. The subspace spanned by the columns of the asset matrix drops rank, which leads to aggregate excess demand being a discontinuous function of spot prices. This, in turn, causes problems for the existence of an equilibrium. The same reasoning applies to the case with price uncertainty. But, following Huang and Wu (1994; 1996), one can consider a model with endogenously chosen options contracts, which can be used to show generic existence of a GEPU.

In Huang and Wu (1996) options are set by an exchange to be at-the-money. This assumption means that the strike price is the date 0 equilibrium price of the underlying commodity futures contract. By the no-arbitrage condition on equilibrium asset prices, the date 0 equilibrium price of the underlying commodity futures contract is a weighted average of the possible equilibrium liquidation values of that futures contract. These weights form what is known as an equivalent martingale measure. There are multiple date 1 equilibrium liquidation values of the underlying futures contract which do not all coincide. In particular, there is a highest and lowest possible date 1 equilibrium value of the underlying futures contract. An at-the-money strike price lies inside the interval formed by the highest and lowest possible liquidation values of the underlying futures contract. This property ensures that generically options do not all pay off zero and therefore that the asset payoff matrix will not drop rank for different equilibrium spot prices.

We note that similar reasoning in the context of price uncertainty can be used to show that a GEPU exists generically for the above described economy with incomplete option markets. With price uncertainty, the no-arbitrage condition still implies a martingale characterization of strike prices, which enables us to prove the generic existence of a GEPU in a way analogous to the method by which Huang and Wu (1996) prove that a GEI (general equilibrium with incomplete markets) with exogenous state uncertainty generically exists. With incomplete options markets, there may exist multiple price equilibria (P1 is true), which was not possible with complete options (Proposition 4). The equilibrium allocation, if it exists, may not be efficient (P2 fails). Options still play the role of providing insurance against price uncertainty, but they cannot eliminate all price risk and D is not a singleton.

5 Price uncertainty without structural knowledge

In this section, we consider the formulation of an economic model where the agents do not have structural knowledge. Given the assumptions of Debreu's (1970) regular economy approach, there are finitely many spot market equilibria for almost all initial endowments. Nonetheless, the set of market equilibria will change when security markets are introduced. This is only natural in a general equilibrium setting, but it means that households may not know at date 0 which are the finitely many equilibrium spot prices that can occur with positive probability at date 1. The agents' date 0 beliefs are not based on structural knowledge of equilibrium prices at date 1. Moreover, actual security markets are not indexed by finitely many possible equilibrium prices. Instead, the most famous example of a price-contingent security, an option, has a payoff that is a continuous function of all underlying prices, not just finitely many equilibrium prices. This motivates the two types of security markets that we discuss below as well as our treatment of probability beliefs over underlying prices. In particular, we do not restrict date 0 beliefs to coincide with date 1 beliefs and date 0 beliefs may or may not have a finite support.

First, we consider the case of infinitely many expected prices. We assume that at date 0, households possess prior probability beliefs with the set C of infinitely many or a continuum of prices as the support. An infinite price state space arises natually when households do not have structural knowledge and cannot rule out any price as impossible, that is, $C = S$. Denote by π^h household h's date 0 subjective probability distribution over S. At date 1, households possess common posterior beliefs over date 1 equilibrium spot prices; they place probability one on the price p^* that is observed. Since $p^* \in s$, their beliefs at date 0 are not contradicted by the data at date 1. This updating of beliefs in light of pub-

licly observable spot market commodity prices becoming available goes to the heart of what price uncertainty means. At date 0, households do not have the necessary information to determine which proper, finite subsets of the unit simplex are possible spot market equilibria. At date 1, all households know the date 1 price. But, they cannot go back in time from date 1 to date 0. This obvious point means that, formally, at date 0, households believe that at date 1 some member of C will prevail, and indeed at date 1, an element of C does realize, namely p^*, which all households can publicly observe. Households in our model are cognizant of this metamorphosis in the sense of holding different rational beliefs at dates 0 and 1. Finally, there are complexity reasons to believe that households at date 0 will not be able to compute the spot equilibrium price vector p^* at date 1. Instead, they merely observe it when date 1 arrives.

Household h selects a consumption plan, $x^h : C \to R$ where, for each $p \in C$, $x^h(p)$ is the quantity of commodities that household h plans to buy if spot price vector p realizes. Let $x = (x^1, \ldots, x^H) \in R^{GH}$ denote a consumption plan allocation, so that $x : C \to R^{GH}$. If $C = S$, commodity plans are members of the function space, $Lp(C)$. Commodity prices p are in the topological dual, $Lq(C)$, where $1/p + 1/q = 1$. The asset structure includes options written on the spot price of the first commodity. The set of possible strike prices, P, is a subset of [0,1]. Let P be specified exogenously to be countable and dense in [0,1]. For example, the set of strike prices can include all rational numbers between 0 and 1.

Each household h also chooses a portfolio $\theta_E^h : P \to R$, for $E = C$ or P. Security portfolios are members of the space $Lp(C)$. As before, $\theta_E = (\theta_E^1, \ldots, \theta_E^H)$ and $\theta = (\theta_C, \theta_p)$. Security prices q are in the topological dual. In order to rule out arbitrage opportunities, we require that security prices be arbitrage-free. This means that no household can find a sequential arbitrage opportunity as defined in Brown and Werner (1994). Our definition of GEPU can be modified when the agents believe that any price in the simplex S is possible ($C = S$).

Definition 6: *A general equilibrium with price uncertainty (GEPU), given a continuum of expected prices, is a set (commonly held across households) of expected equilibrium spot prices $B = (S, p^*)$, a commodity spot market price vector p^*, a consumption plan allocation $x(p)$, a security price vector q, and a portfolio allocation θ such that q is a no-arbitrage security price, and:*

For all h,

$$\left(x^h(p), \theta^h(k)\right) \text{ maximizes } Eu^h\left(x^h(p)\right) = \int_s u^h\left(x^h\right)\pi^h(p)dp$$

subject to the budget constraints:

$$\sum_{k \in P} \sum_E q_E(k)\theta_E^h(k) = 0 \ (\textit{option budget constraint}) \tag{5.1}$$

$$px^h(p) = \sum_{k \in P} \sum_E V_E(p, k)\theta_E^h(k) + pe^h \textit{ for each } p \in S$$
$$(\textit{ex ante budget constraint})$$

and

$$\sum_h \theta_E^h(K) = 0 \textit{ for } E = C,\ P \textit{ and all } K \in P \tag{5.2}$$
$$(\textit{option market clearing})$$

$$\sum_h x^h(p) = \sum_h e^h \textit{ for each } p \in S \tag{5.3}$$
$$(\textit{ex ante commodity market clearing})$$

$$\sum_h \theta_E^h(K) = 0 \ (\textit{option market clearing}) \tag{5.4}$$

Furthermore, $x^h(p^*)$ maximizes $u^h(x^h(p^*))$ subject to:

$$p^* x^h(p) = p^* e^h + \sum_{k \in P} \sum_E V_E(p^*, k)\theta_E^h(k) \tag{5.5}$$
$$(\textit{ex post budget constraint})$$

and

$$\sum_h x^h(p^*) = \sum_h e^h \tag{5.6}$$
$$(\textit{ex post commodity market clearing})$$

There is a limited amount of work about asset market equilibria when the state space is infinite. Brown and Ross (1991) show how the Ross (1976) result can be extended to an infinite state space. Aliprantis and Brown (1994) answer the question of what asset market in completeness means when there is a continuum of states. If options complete markets in the sense of Aliprantis and Brown (1994), then existence of market

equilibrium follows by applying the standard infinite dimensional space results. By applying Aliprantis and Brown (1994), one may show that a GEPU exists if the set of marketed assets M is the Lp-closure of the space of European call options.

Our model differs from Svensson's (1981) model of Arrow–Debreu markets for commodities contingent on any price in the simplex because the possibility of short sales of securities in our model implies that households face a commodity space that may not be bounded below. Svensson's results are not applicable to financial markets. Our model also differs from Henrotte's (1992) analysis because he studies a three-period model in which there are infinitely many derivative securities traded in period 1, finitely many primitive securities traded in period 2, and consumption occurs in period 3. Henrotte's results are based on a closedness condition utilized by Mas-Colell (1986) and Aliprantis, Brown, and Burkinshaw (1989). This means that the utility possibility set is a closed subset of R^H, which also awaits verification in a framework of incomplete options. Whether proof of existence can be shown for a continuum of fixed possible endogenous price states is unclear, because there are several well-known unsolved problems with incomplete security markets for infinite dimensional spaces, as discussed in Mas-Colell and Zame (1991).

Next consider the case of finitely many expected prices. When the support of agents' date 0 beliefs is a proper subset C of the price simplex S, there is no guarantee that the realized price actually belongs to C. The agent's beliefs can therefore be contradicted with observed data. This is especially true when there are only finitely many expected prices. Kurz and Wu (1996) work with the case of a finite support of beliefs and show that these beliefs with some rationality restrictions can be made compatible with the observed data.

When agents do not posess structural knowledge, the temporary equilibrium approach taken by Svensson (1981) and Henrotte (1992) does not use any rationality conditions and has to deal with a space of the order of a continuum. As discussed before, it entails some technical complexities. On the other hand, the rational expectations approach covered in the last section endows the agents with an unrealistically high capability of computing economic equilibria. The approach by Kurz and Wu (1996) formulates an equilibrium model with rational beliefs to account for endogenous price uncertainty and to establish the existence and efficiency of market equilibrium when there is a complete set of options markets. They also construct a finite "price state space" which includes exogenous states and the states of diffuse beliefs. However, the case with incomplete options markets remains unresolved as yet.

6 Conclusion

We have demonstrated how to extend the traditional Arrow–Debreu general equilibrium analysis of an economy with exogenous state uncertainty to deal with endogenous price uncertainty. In his Nobel lecture, Arrow (1974) stated that he viewed the model of general equilibrium with uncertainty not only as a normative ideal for public policy to strive for, but also as a description of existing reality. As Shiller (1993) pointed out, the focus of economic theory should be on an economy's largest risks, namely fluctuations in aggregate income and prices. Such important fluctuations in the economy cannot be solely attributed to exogenous shocks.

It has been two decades since Kurz's (1974) call to model endogenous uncertainty. Kurz himself and others have recently made progress in that endeavor, but much remains to be done. The contribution of this essay is to provide a particular and natural formulation of a type of endogenous uncertainty, namely, price uncertainty and indicate the subtle issues in showing existence of an equilibrium and establishing its properties in the face of endogenous price uncertainty. We also examine the role of options in dealing with endogenous price uncertainty. The attempt to understand the full implications of endogenous uncertainty presents a very interesting challenge for economic theory (and theorists) for decades to come.

References

Aliprantis, C. D., and D. J. Brown. 1994. Existence of competitive equilibria in markets for derivative securities. Working paper, Stanford University.

Aliprantis, C. D., D. J. Brown, and O. Burkinshaw. 1989. Existence and optimality of competitive equilibria. New York: Springer-Verlag.

Arrow, K. J. 1953. The role of securities in the optimal allocation of risk-bearing. *Econometrie*. Translated and reprinted in *Review of Economic Studies* (1964)31:91–6.

Arrow, K. J. 1974. General economic equilibrium: Purpose, analytic techniques, collective choice. *American Economic Review*, 61:253–72.

Arrow, K. J. 1978. The future and the present in economic life. *Economic Inquiry*, 16:157–70.

Arrow, K. J. 1982. Risk perception in psychology and economics. *Economic Inquiry*, 20:1–9.

Arrow, K. J. 1986. Rationality of self and others in an economic system. *Journal of Business*, 59:S385–99.

Arrow, K. J. 1991. Economic forecasting. In *New Technologies and the Future of Food and Nutrition*, ed. G. E. Gaull. New York: Wiley.

Arrow, K. J. 1994. General equilibrium under uncertainty: Survey and problems. Presented at SITE (Stanford Institute of Theoretical Economics).

Arrow, K. J., and G. Debreu. 1954. Existence of equilibrium for a competitive economy. *Econometrica*, 22:265–90.
Aumann, R. J., and A. Brandenburger. 1991. Epistemic conditions for Nash equilibrium. Working Paper No. 91-042, Harvard Business School.
Bewley, T. 1972. Existence of equilibrium in economies with infinitely many commodities. *Journal of Economic Theory*, 4:514–40.
Brown, D., and S. A. Ross. 1991. Spanning, valuation and options. *Economic Theory*, 1:3–12.
Brown, D. J., and J. Werner. 1995. Arbitrage and existence of equilibrium in infinite asset markets. *Review of Economic Studies*, 62:101–14.
Cass, D., G. Chichilnisky, and H. M. Wu. 1966. Individual risk and mutual insurance. *Econometrica*, 64:333–41.
Chichilnisky, G. 1992. General equilibrium with endogenous uncertainty. Working paper, Columbia University; this volume.
Chichilnisky, G. 1994. Markets with endogenous uncertainty: Theory and policy. Presented at SITE, Stanford University, August, *Theory and Decision*, 41: 99–131, 1996.
Chichilnisky, G., J. Dutta, and G. M. Heal. 1991. Price uncertainty and incomplete markets. Columbia Business School working paper.
Chichilnisky, G., G. M. Heal, and D. P. Tsomocos. 1995. Option values and endogenous uncertainty in ESOPs, MBOs, and asset-backed securities. *Economics Letters*, 48(3–4):379–88.
Chichilnisky, G., and H. M. Wu. 1992. Financial innovation and endogenous uncertainty in incomplete asset markets. Working paper; SITE Technical Report No. 50, Stanford.
Debreu, G. 1959. *Theory of Value*. New Haven: Yale University Press.
Debreu, G. 1970. Economies with a finite set of equilibria. *Econometrica*, 38:387–92.
Duffie, D. 1991. The theory of value in security markets. In *Handbook of Mathematical Economics*, Vol. 4, ed. W. Hildenbrand and H. Sonnenschein. Amsterdam: North-Holland, pp. 1614–82.
Hahn, F. 1992. On Arrow securities. Working paper, Cambridge University.
Hahn, F. 1994. A remark on incomplete market equilibrium. Working paper, Cambridge University; this volume.
Harkasson, N. H. 1978. Welfare of options and supershares. *Journal of Finance*, 33:759–76.
Henrotte, P. 1992. Existence and optimality of equilibria in markets with tradeable derivative securities. SITE Technical Report No. 48, Stanford University.
Hicks, J. 1946. *Value and Capital*. Oxford: Clarendon Press.
Huang, C. 1993. Time and uncertainty. Presented at the General Equilibrium Theory 40th Anniversary Conference at CORE, Louvain-la-Neuve, Belgium, June.
Huang, P. H., and H. M. Wu. 1994. Competitive equilibrium of incomplete markets for securities with smooth payoffs. *Journal of Mathematical Economics*, 23:219–34.
Huang, P. H., and H. M. Wu. 1996. General equilibrium with incomplete markets of European options. Working paper, National Taiwan University.

Geanakoplos, J., and H. Polemarchakis. 1996. Existence, regularity, and constrained suboptimality of competitive allocations when markets are incomplete, in *Uncertainty, Information, and Communication: Essays in Honor of Kenneth Arrow*, Vol. 3, ed. W. P. Heller, R. M. Ross, and D. A. Starrett. Cambridge: Cambridge University Press.
Kreps, D. 1992. Static choice in the presence of unforeseen contingencies. In *Economic Analysis of Markets and Games: Essays in Honor of Frank Hahn*, ed. P. Dasgupta, D. Gale, O. Hart, and E. Maskin. Cambridge, MA: MIT Press, pp. 258–81.
Kurz, M. 1974. The Kesten–Stigum model and treatment of uncertainty in equilibrium theory. In *Essays on Economic Behavior Under Uncertainty*, ed. M. S. Balch, D. L. McFadden, and S. Y. Wu. Amsterdam: North-Holland, pp. 389–99.
Kurz, M. 1994a. On the structure and diversity of rational beliefs. *Economic Theory*, 4:877–900.
Kurz, M. 1994b. On rational belief equilibrium. *Economic Theory*, 4:859–76.
Kurz, M. 1996. Rational beliefs and endogenous uncertainty: Introduction. *Economic Theory*, 8:383–97.
Kurz, M., and H. M. Wu. 1996. Endogenous uncertainty in a general equilibrium model with price-contingent contracts. *Economic Theory*, 8:461–88.
Mas-Colell, A. 1986. The price equilibrium existence problem in topological vector lattices. *Econometrica*, 54:1039–53.
Mas-Colell, A. 1992. Three observations on sunspots and asset redundancy. In *Economic Analysis of Markets and Games, Essays in Honor of Frank Hahn*, ed. P. Dasgupta, D. Gale, O. Hart, and E. Maskin. Cambridge, MA: MIT Press, pp. 465–74.
Mas-Colell, A., and W. R. Zame. 1991. Equilibrium theory in infinite dimensional spaces. In *Handbook of Mathematical Economics*, Vol. 4, ed. W. Hildenbrand and H. Sonnenschein. Amsterdam: North-Holland, pp. 1835–98.
Negishi, T. 1960. Welfare economics and existence of an equilibrium for a competitive economy. *Metroeconomica*, 12:92–7.
Polemarchakis, H. M., and B. Ku. 1990. Options and equilibrium. *Journal of Mathematical Economics*, 19:107–12.
Ross, S. 1976. Options and efficiency. *Quarterly Journal of Economics*, 90:85–9.
Savage, L. 1954. *The Foundations of Statistics*. New York: Wiley.
Shiller, R. J. 1981. Do stock prices move too much to be justified by subsequent changes in dividends? *American Economic Review*, 71:421–36.
Shiller, R. J. 1993. *Macro Markets: Creating Institutions for Managing Society's Largest Economic Risks*. Oxford: Oxford University Press.
Svensson, L. E. O. 1981. Efficiency and speculation in a model with price-contingent contracts. *Econometrica*, 49:131–51.