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Competitive equilibrium of incomplete markets for securities with smooth payoffs*

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We study an economy with incomplete markets for securities, whose payoffs are smooth, possibly nonlinear, functions of spot commodity prices. We prove competitive equilibria exist for an open and dense set in the product of the Euclidean topology and the Whitney C^1 topology on the space of initial endowments and security payoff functions.

Key words: General equilibrium of incomplete security markets; Smooth payoffs; Generic existence of competitive equilibria

JEL classification: D52; G10

1. Introduction

Recently, there has been renewed interest about incomplete security markets. Hart (1975) provided an example of nonexistence of competitive equilibria for incomplete real securities, which are claims to commodity bundles. Given this finding, Duffie and Shafer (1985, 1986a, b) prove generic existence of competitive equilibria of incomplete real security markets. Husseini et al. (1990), Geanakoplos and Shafer (1990), and Hirsch et al. (1990) show the classical Brouwer or Kakutani fixed point theorems do not suffice for proving existence of competitive equilibria in the presence of incomplete real security markets. They offer new types of general fixed point

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results that can be used when there are incomplete real security markets. In all existing applications of these fixed point theorems, the value of payoffs to real securities is an affine function of spot commodity prices. The purpose of this paper is to provide sufficient conditions for the generic existence of competitive equilibria if security payoffs are smooth (possibly nonlinear) functions of spot commodity prices.

We prove that smoothness of preferences and security payoff functions ensures generic existence of competitive equilibria when there are incomplete markets for securities with nonlinear payoffs. Smoothness of security payoff functions permits the application of transversality theorems. A novelty of our approach is that we vary security payoff functions in the Whitney C^1 topology, instead of varying finite dimensional security structures, as in the existing models, in order to guarantee that distinct spot commodity prices ensure the matrix of security payoff values has maximal rank. In our model, the matrix of security payoff values can also drop rank if the equilibrium spot commodity prices have less distinct values across states than the number of securities. Another new feature of our approach is that we perturb initial endowments to ensure that generically spot commodity prices are distinct across states. We prove that competitive equilibria exist for a set of initial endowments and security payoff functions that is open and dense in the product of the Euclidean and the Whitney C^1 topologies.

Smoothness of payoff functions is crucial since Polemarchakis and Ku (1990) have constructed a counterexample to existence of competitive equilibria of incomplete markets for nonsmooth security payoff functions, that are only continuous, such as those for European options. Their example is robust in the space of initial endowments and strike prices.[1] We extend our main result to incomplete asset market economies where investors are endowed with positive amounts of those securities. This occurs in models of incomplete stock markets, such as Duffie and Shafer (1986b), Geanakoplos et al. (1990), or Magill and Quinzii (1989). They demonstrate the existence of competitive equilibria for generic initial endowments of consumers and generic translations of production sets.

Our generic existence result contrasts with full existence results involving financial securities, as in Arrow (1953), Cass (1984), Werner (1985), and Duffie (1987) or involving real numeraire securities, as in Geanakoplos and Polemarchakis (1986). Excellent expositions of the recent work on incomplete security markets can be found in Duffie (1988, 1991), Duffie et al. (1988),

[1]This problem has been dealt with by Krasa (1989), Krasa and Werner (1991) and Huang and Wu (1994). Krasa introduces a condition on the distribution of the aggregate level of endowments across the states of nature and showed that under that condition, the fraction of economies with a competitive equilibrium in a model of European commodity options approaches one. Krasa and Werner model nominal asset options having strike prices that are both given exogenously and varied by normalizing prices. Huang and Wu (1994) analyze options written on real assets. They show generic existence when strike prices are set equal to the endogenously determined prices of the underlying assets.

Geanakoplos (1990), Magill and Shafer (1991), or Marimon (1987). A crucial difference between real versus financial or real numeraire securities is that while the subspace of income transfers achievable with financial or real numeraire securities is exogenous and independent of spot commodity prices, the subspace of income transfers achievable with real securities is endogenous and dependent on spot commodity prices. We obtain a generic existence result because the subspace of achievable income transfers in our model is also dependent on spot commodity prices.

Section 2 presents our general model and provides the definition of a competitive equilibrium. Section 3 defines two related equilibrium concepts, namely pseudo-equilibria and regular pseudo-equilibria. Section 4 presents our main generic existence result. Section 5 extends that result to positive initial security endowments. Section 6 collects proofs of all results.

2. Model

Consider a two-period economy with G physical goods, S states of nature, and $N=G(1+S)$ spot good markets, G spot good markets in the initial period and in each of the S states of nature. There are H households with initial endowments in period zero and state-contingent endowments in period one: $e^h=(e_0^h, e_1^h(s))\in\mathbb{R}^N_{++}$. We assume that households' preferences are differentiable, strictly monotone, strictly convex, and satisfy the usual boundary condition [Debreu's (1970, 1972) smooth preference assumptions].

We denote the N dimensional vector of period zero and period one state-contingent spot good markets prices by p, which can be divided into its period zero and period one components: $p=(p_0, p_1(s))\in\mathbb{R}^N_+$. We choose good G as numeraire, so that $p_0^G=1$ and $p_1^G(s)=1$ for all $s=1,\dots,S$. Consumption bundles can also be divided into their period zero and period one components: $x^h=(x_0^h, x_1^h(s))\in\mathbb{R}^N_+$.

We introduce an incomplete set of $M(<S)$ securities whose payoffs depend on spot good prices and the states in the first period. Securities are traded in period zero before the state of nature is revealed. One unit of the jth security for $j=1,\dots,M$ is a contract delivering $a_{js}(p_1(s))$ units of the numeraire good G in state s for $s=1,\dots,S$. The possible dependence of a_{js} on the entire G-dimensional vector $p_1(s)$ allows the payoff security j to depend on an index of spot commodity prices or the vector of goods a firm produces in the case of dividends paid by equity. We write the collection of S state-dependent payoff functions as $a_j(p_1)=(a_{js}(p_1(s)))_{s=1}^S$, where $a_j(p_1)$ maps $\mathbb{R}^{SG}_+$ to $\mathbb{R}^S_+$. Households are allowed to trade these securities, which are assumed to be in zero net supply. Bonds, futures, and options are examples of securities in zero net supply. We assume that households are endowed with zero amounts of these securities. In section 5, we allow the households to be endowed with positive shares of securities, such as the case with stocks or equity holdings.

Let $a(p_1)=(a_1(p_1),\ldots,a_M(p_1))$ be the *function* mapping $\mathbb{R}_+^{SG}$ to $\mathbb{R}_+^{SM}$, whose jth coordinate is the function $a_j(p_1)$. In the interests of brevity, we write $a_j(p)$ instead of $a_j(p_1)$. We assume that security payoff functions satisfy

Assumption SP (*smooth payoff functions*). The function $a_{js}:\mathbb{R}_+^G\to\mathbb{R}_+$ is continuously differentiable for $j=1,\ldots,M$. In other words, $a(p)$ is an element of the function space $C^1(\mathbb{R}_+^{SG},\mathbb{R}_+^{SM})$ of continuously differentiable functions mapping SG-dimensional nonnegative Euclidean space into SM-dimensional nonnegative Euclidean space.

Assumption SP permits the application of transversality theorems. Our model applies to primitive real securities or derivative securities with linear payoffs because the set of linear payoff functions is a subset of $C^1(\mathbb{R}_+^{SG},\mathbb{R}_+^{SM})$. Note that if security payoffs are required to satisfy additional restrictions such as strict monotonicity, as long as those conditions define an open set in the Whitney C^1 topology, our genericity proof can be modified to deal with such restrictions.

If a particular security can be replicated by a portfolio of the remaining securities, then that security is redundant. It can be left out of the set of non-redundant securities whose prices we are trying to determine because its price is already completely determined by arbitrage. So, without loss of generality, we assume that none of the security payoff functions is a linear combination of the remaining security payoff functions *for all* possible $p_1(s)$.

An economy is characterized by utility functions, initial commodity and security endowments, and security payoff functions: $E((u^h,e^h),a(p))$. Let θ^h denote the portfolio of security choices by household h with $\theta^h=(\theta_j^h)\in\mathbb{R}^M$. Let $q=(q_j)\in\mathbb{R}^M$ denote security prices.

Definition 1. A competitive equilibrium for an economy $E((u^h,e^h),a(p))$ is a pair $((x^h,\theta^h),(p,q))$ such that:

$$(x^h,\theta^h)=\arg\max u^h(x^h) \quad \text{subject to}$$

$$p_0(x_0^h-e_0^h)+\sum_{j=1}^M q_j\theta_j^h=0, \quad \text{and} \tag{2.1}$$

$$p_1(s)(x_1^h(s)-e_1^h(s))=\sum_{j=1}^M a_{js}(p(s))\theta_j^h \quad \text{for} \quad s=1,\ldots,S; \tag{2.2}$$

$$\sum_h x^h=\sum_h e^h; \quad \text{and} \tag{2.3}$$

$$\sum_h \theta^h=0. \tag{2.4}$$

Let $V(p)$ be the S and M dimension matrix of state-dependent security payoffs. Notice the period one budget constraints can be rewritten as $p_1\Box(x_1^h - e_1^h) \in \langle V(p)\rangle$, where $\langle V(p)\rangle$ is the subspace of R^S spanned by the M columns of $V(p)$ and we use the notation of Duffie and Shafer (1985), $p_1\Box(x_1^h - e_1^h) = (p_1(s)(x_1^h(s) - e_1^h(s)))_{s=1}^S$. Define the $S+1$ by M matrix $W(p,q)$, with its first row, the vector $-(q)$, and its next S rows, $V(p)$. We rewrite eqs. (2.1)–(2.2) as $p(x^h - e^h) = W(p,q)\theta^h$. We define the subspace of income transfers in $\mathbb{R}^{S+1}$ generated by the columns of $W(p,q)$ to be $\langle W(p,q)\rangle = \{\tau \in \mathbb{R}^{S+1} \mid \exists\theta \in \mathbb{R}^M$ with $\tau = W(p,q)\theta\}$; and the orthogonal (dual) subspace of state prices, $\langle W(p,q)\rangle^\perp = \{\alpha \in \mathbb{R}^{S+1} \mid \alpha W(p,q) = 0\}$.

We introduce the concept of no-arbitrage security prices:[2]

Definition 2. q is a no-arbitrage security price if there does not exist a portfolio θ in $\mathbb{R}^M$ with a semipositive return $W(p,q)\theta \geqq 0$.

Clearly, the households' optimization problem has a finite maximum if and only if q is a no-arbitrage vector of security prices. This lemma characterizes a no-arbitrage security price vector by showing the existence of positive state prices, β.

Lemma 1. If q is a no-arbitrage security price vector, then there exists $\beta = (\beta_0, \beta_1, \ldots, \beta_S) \in \mathbb{R}_{++}^{S+1}$, such that

$$q_j = \sum_{s=1}^{S} \beta_s a_{js}(p(s)). \tag{2.5}$$

3. Related equilibrium concepts

In this section, we introduce two other concepts of equilibria besides the competitive one, namely that of pseudo-equilibria and regular pseudo-equilibria, which are used in the proof of generic existence of competitive equilibria.

First, we eliminate the security prices by noting that the period zero budget constraint (2.1) can be rewritten by using the no-arbitrage condition (2.5):

$$p_0[x_0^h - e_0^h] + \sum_{j=1}^{M}\left[\sum_{s=1}^{S} \beta_s a_{js}(p(s))\right](\theta_j^h) = 0.$$

Using the period one budget constraint (2.2), this can be rewritten

[2] We follow the notational convention that $x \geqq 0$ means $x_i \geqq 0$ for all i and $x_i > 0$ for some i.

$$p_0[x_0^h - e_0^h] + \sum_{s=1}^{S} \beta_s\{p_1(s)[x^h(s) - e^h(s)]\} = 0.$$

Following Husseini et al. (1990), consider the price simplex $\Delta_+^{N-1} = \{p \in \mathbb{R}_+^N \mid \sum_{i=1}^N p^i = 1\}$. We can use the homogeneity (of degree zero) property of the period one budget constraint (in the vector of period one spot good prices) to rescale spot good prices so that $\beta_s p(s)$ can be replaced by $p(s)$ without affecting the budget constraint.[3] So, we can rewrite the above equation as $p_0[x_0^h - e_0^h] + \sum_{s=1}^S p(s)[x^h(s) - e^h(s)] = 0$, or $p(x^h - e^h) = 0$. After we have found a competitive equilibrium, we apply the homogeneity property of the period one budget constraint (in the vector of period one spot good prices) to rescale spot good prices so that $p^G(s) = 1$ for all $s = 1, \ldots, S$.

So as to define pseudo-equilibria and regular pseudo-equilibria, we replace $\langle V(p) \rangle$, the subspace of actual income transfers achievable by trading in securities with a trial subspace of feasible income transfers. In order to have a sufficiently rich family of subspaces from which to find an equilibrium one we require a convenient way to vary these subspaces.[4] We follow Husseini et al. (1990) and vary M-dimensional subspaces of $\mathbb{R}^S$ by studying the Stiefel manifold: $O^{S,S-M} = \{Q \in \mathbb{R}^{(S-M)S} \mid QQ^{\mathrm{T}} = I\}$ of all $(S-M)$ by S orthonormal matrices Q. If $Q \in O^{S,S-M}$, the span of the columns of Q transpose, $\langle Q^{\mathrm{T}} \rangle$, is a subspace of $\mathbb{R}^S$, that has dimension $(S-M)$. By an orthogonal decomposition of $\mathbb{R}^S = \langle Q^{\mathrm{T}} \rangle \oplus \langle Q^{\mathrm{T}} \rangle^{\perp}$, we get the M-dimensional subspace, $\langle Q^{\mathrm{T}} \rangle^{\perp}$. Since there are many matrices $Q \in O^{S,S-M}$ that generate the same subspace we observe that if $O(S-M)$ is the orthogonal group of $(S-M)$ by $(S-M)$ orthogonal matrices, then $\forall g \in O(S-M)$, $\langle Q^{\mathrm{T}} \rangle = \langle (gQ^{\mathrm{T}}) \rangle$. We use the Stiefel manifold representation for subspaces to replace the subspace of income transfers $\langle V(p) \rangle$ achievable with security trading by a trial subspace $\langle Q^{\mathrm{T}} \rangle^{\perp}$ in the budget constraints of households 2 though H. Following the literature, there is no loss of generality if household one is not constrained by security markets [see, for example, Cass (1984) and Geanakoplos (1990)]. Define the budget correspondence

$$B: \Delta_+^{N-1} \times O^{S,S-M} \times \mathbb{R}_{++}^N \to \mathbb{R}_+^N \quad \text{for} \quad h = 2, \ldots, H$$

as follows:

$$B(p, Q; e^h) = \{x \in \mathbb{R}_+^N \mid p(x^h - e^h) = 0 \text{ and } p(1) \square [x^h(1) - e^h(1)] \in \langle Q^{\mathrm{T}} \rangle^{\perp}\}. \tag{3.1}$$

[3]Although the security payoffs are nonlinear as functions of period one spot good prices with respect to a numeraire, namely, the last good, security payoffs are homogeneous of degree zero as functions of all good prices.

[4]There is an equivalent way introduced by Duffie and Shafer (1985) to vary M-dimensional subspaces of $\mathbb{R}^S$. This involves studying the Grassmanian manifold $G^{M,S}$ of all M-dimensional linear subspaces of $\mathbb{R}^S$.

Then, because $\forall g \in O(S-M), \langle Q^{\mathrm{T}} \rangle = \langle (gQ^{\mathrm{T}}) \rangle$; $B(p,Q;e^h)$ is $O(S-M)$-invariant: $\forall g \in O(S-M)$, $B(p,gQ;e^h) = B(p,Q;e^h)$.

Definition 3. A pseudo-equilibrium for the economy $E((u^h,e^h), a(p))$ over the Stiefel manifold is a three-tuple $(x,p,Q) \in \mathbb{R}_+^{HN} \times \Delta_+^{N-1} \times O^{S,S-M}$, consisting of an allocation of goods, prices for goods, and a trial subspace of income transfers such that:

$$x^1 = \arg\max u^1(x^1) \quad \text{subject to} \quad p(x^1 - e^1) = 0, \tag{3.2}$$

$$x^h = \arg\max u^h(x^h) \quad \text{subject to} \quad x^h \in B(p,Q;e^h) \quad \text{for} \quad h = 2,\dots,H; \tag{3.3}$$

$$\sum_h x^h = \sum_h e^h; \quad \text{and} \tag{2.2}$$

$$\langle Q^{\mathrm{T}} \rangle^{\perp} \supset \langle V(p) \rangle. \tag{3.4}$$

Existence of a pseudo-equilibrium is demonstrated by applying Husseini et al. (1990) fixed point theorem, that is also proved by Hirsch et al. (1990). We state this result in our notation:

Husseini, Lasry and Magill Fixed Point Theorem. Let H^{N-1} be an $(N-1)$-dimensional affine subspace, $H^{N-1} \supset D$, a compact convex subset with non-empty relative interior. Let (Φ, Ψ) be continuous functions $\Phi: D \times O^{S,S-M} \to H^{N-1}$ and $\psi: D \times O^{S,S-M} \to (\mathbb{R}^{S-M})^M$ such that $\forall Q \in O^{S,S-M}$,

$$D \supset \Phi(\partial D, Q), \tag{3.5}$$

$$\Phi(p,gQ) = \Phi(p,Q) \quad \forall g \in O(S-M) \text{ and } \forall (p,Q) \in D \times O^{S,S-M}; \text{ and} \tag{3.6}$$

$$\psi(p,gQ) = g\psi(p,Q) \quad \forall g \in O(S-M) \text{ and } \forall (p,Q) \in D \times O^{S,S-M}. \tag{3.7}$$

Then, there exists $(p,Q) \in D \times O^{S,S-M}$ such that $\Phi(p,Q) = p$ and $\psi(p,Q) = 0$.

Finally, we define[5]

Definition 4. A pseudo-equilibrium (x,p,Q) is regular if $\langle Q^{\mathrm{T}} \rangle^{\perp} = \langle V(p) \rangle$.

Following Husseini et al. (1990), we have

Theorem 1. Any economy $E((u^h,e^h), a(p))$ has a pseudo-equilibrium.

[5]A regular pseudo-equilibrium is what Duffie and Shafer (1985) term an effective equilibrium.

4. Generic existence of competitive equilibria

We show that in the space of security payoff functions and initial endowments, generically a pseudo-equilibrium is regular. Our topological notion of genericity generalizes the measure-theoretic notion of genericity in a finite dimensional space to an infinite dimensional space. Our proof of the existence of pseudo-equilibria applies equally well to non-smooth security payoff functions; but, the genericity result does not apply to securities without smooth payoff functions. Let us explain why intuitively the proof works in the case of C^1 payoff functions. A pseudo-equilibrium is not regular if the security payoff matrix drops rank. This might happen since for a particular set of $p(s)$ values, either (1) the columns of the matrix $V(p)$ are linerly dependent or (2) $p(s)$ has less than M distinct values across the S states. This means that M rows of the matrix $V(p)$ are linearly dependent and hence the row rank of $V(p)$ is less than M. If the first possibility happens, we can perturb the function $a_{js}(p)$ to another C^1 function that is close in the Whitney C^1 topology so that linear independence is restored. We also show that in the space of initial endowments generically the second possibility does not happen. We actually prove a stronger result, namely that for generic initial endowments, $p(s)$ is distinct across all of the S states, not just for M of them. We do this by proving that generically in the space of initial endowments, the negation of this does not hold.

Theorem 2. There is an open and dense set Ω in $C^1(\mathbb{R}_+^{SG}, \mathbb{R}_+^{SM}) \times \mathbb{R}_{++}^{HN}$, such that for every $(a(p), e)$ in Ω, every pseudo-equilibrium of an economy $E((u^h, e^h), a(p))$ is regular.

Following Duffie and Shafer (1985) or Husseini et al. (1990) we have this result:

Theorem 3. There is an open and dense set Ω in $C^1(\mathbb{R}_+^{SG}, \mathbb{R}_+^{SM}) \times \mathbb{R}_{++}^{HN}$, such that for any economy $E((u^h, e^h), a(p))$ having $(a(p), e)$ in Ω, a competitive equilibrium exists.

This theorem extends the current results on the generic existence of competitive equilibrium for incomplete real securities markets. In existing models, the security returns matrix $V(p)$ has typical element, $[V(p)]_{sj} = p(s)c_j$, the spot market value of the commodity bundle c_j promised by security j. In our model, the security returns matrix $V(p)$ has typical element, $[V(p)]_{sj} = a_{js}(p(s))$, where a_{js} is a state-dependent continuously differentiable function of spot commodity prices.

5. Positive initial security endowments

Consider the case where households own initial endowments of securities:

$\gamma_0^h = (\gamma_{j0}^h) \in \mathbb{R}_+^M$. By virtue of the fact these are shares, $\sum_h \gamma_{j0}^h = 1$ for $j = 1, \ldots, M$. An economy is characterized by utility functions, initial commodity endowments, asset endowments, and asset payoff functions: $E((u^h, e^h, \gamma_0^h), a(p))$. Let γ^h denote the portfolio of asset choices by household h with $\gamma^h = (\gamma_j^h) \in \mathbb{R}^M$. If we consider the net asset trade $\theta^h = \gamma^h - \gamma_0^h$, then $\sum_h \theta^h = 0$ because $\sum_h \gamma_j^h = 1$. Notice that the period one budget constraint $p_1(s)(x_1^h(s) - e_1^h(s)) = \sum_{j=1}^M a_{js}(p(s))\gamma_j^h$ for $s = 1, \ldots, S$ can be written as

$$p_1(s)(x_1^h(s) - e_1^h(s)) - \sum_{j=1}^M a_{js}(p(s))\gamma_{j0}^h = \sum_{j=1}^M a_{js}(p(s))\theta_j^h. \tag{5.1}$$

Let us define

$$w_1^h(a_{js}(p(s)), \gamma_0^h, s) = (w_1^{hg}(a_{js}(p(s)), \gamma_0^h, s))_{g=1}^G, \tag{5.2}$$

where

$$w_1^{hg}(a_{js}(p(s)), \gamma_0^h, s) = e_1^{hg}(s) + \left[\sum_{j=1}^M a_{js}(p(s))\gamma_{j0}^h\right] \Big/ [(G)(p_1^g(s))]. \tag{5.3}$$

Then we can rewrite the period one budget constraint as

$$p_1(s)(x_1^h(s) - w_1^h(a_{js}(p(s)), \gamma_0^h, s)) = \sum_{j=1}^M a_{js}(p(s))\theta_j^h \quad \text{for} \quad s = 1, \ldots, S. \tag{5.4}$$

Now, we can define

Definition 5. A competitive equilibrium for an economy with initial security endowments $E((u^h, e^h, \gamma_0^h), a(p))$ is a pair $((x^h, \gamma^h), (p, q))$ such that:

$$(x^h, \gamma^h) = \arg\max u^h(x^h) \quad \text{subject to}$$

$$p_0(x_0^h - e_0^h) + \sum_{j=1}^M q_j \theta_j^h = 0, \quad \text{and} \tag{2.1}$$

$$p_1(s)(x_1^h(s) - w_1^h(a_{js}(p(s)), \gamma_0^h, s)) = \sum_{j=1}^M a_{js}(p(s))\theta_j^h \quad \text{for} \quad s = 1, \ldots S; \tag{5.4}$$

$$\sum_h x^h = \sum_h e^h; \quad \text{and} \tag{2.3}$$

$$\sum_h \theta^h = 0. \tag{2.4}$$

Note the difference between Definition 5 and Definition 1. The economy is characterized by one additional parameter, γ_0^h, the initial holdings of securities. Comparing eq. (5.4) with eq. (2.2), we notice the only difference is the replacement of commodity endowment, $e_1^h(s)$ in eq. (2.2) by total endowment, $w_1^h(a_{js}(p(s)), \gamma_0^h, s)$ in eq. (5.4). But, $a_{js}(p)$ and γ_0^h characterize an economy in the same manner that e_1^h does. Thus, we have this result:

Theorem 4. There is an open and dense set Ω in $C^1(\mathbb{R}_+^{SG}, \mathbb{R}_+^{SM}) \times \mathbb{R}_{++}^{HN}$, such that for any $E((u^h, e^h, \gamma_0^h), a(p))$ having $(a(p), e)$ in Ω, a competitive equilibrium exists.

6. Proofs

Proof of Lemma 1. This is a consequence of a corollary of the Minkowski Farkas theorem, which states that for any $(S+1)$ by M matrix W, one of these conditions holds: $\langle W(p,q)\rangle \cap \mathbb{R}_+^{S+1} \backslash 0$ is non-empty or $\langle W(p,q)\rangle^{\perp} \cap \mathbb{R}_{++}^{S+1}$ is non-empty. This means that either $\exists\theta \in \mathbb{R}^M$ such that $W\theta \geqq 0$ or $\exists\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_S) \in \mathbb{R}_{++}^{S+1}$ such that $\alpha W = 0$. Since q is assumed to be a no-arbitrage asset price, there does not exist $\theta \in \mathbb{R}^M$ such that $W\theta \geqq 0$. Therefore, $\exists\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_S) \in \mathbb{R}_{++}^{S+1}$ such that $\alpha W = 0$, or

$$0 = -\alpha_0 q_j + \sum_{s=1}^{S} \alpha_s a_{js}(p(s)). \tag{6.1}$$

Notice that we can divide both sides of eq. (6.1) by the scalar α_0, since that is guaranteed to be positive. This results in the desired β's and eq. (2.5).

Proof of Theorem 1. We apply the Husseini–Lasry–Magill fixed point theorem with $D = \Delta_+^{N-1}, H^{N-1} = \{p \in \mathbb{R}^N \,|\, \sum_{i=1}^N p^i = 1\}$, Φ equal to a price adjustment function that is a modification of the aggregate excess demand function as constructed in Husseini et al. (1990), and $\psi(p,Q) = QV(p)$. It can be verified as in Husseini et al. (1990) that (3.5)–(3.7) are satisfied, so that a fixed point (p^*, Q^*) exists such that $\Phi(p^*, Q^*) = p^*$ and $\psi(p^*, Q^*) = 0$. This fixed point is a pseudo-equilibrium. Q.E.D.

Proof of Theorem 2. By the smooth preference assumption, the solution to each household's utility maximization problem exists, is unique, and results in a system of individual demand functions of spot commodity prices. Consider the modified aggregate excess demand function $z^*: \mathbb{R}_{++}^N \times O^{S,S-M} \times C^1(\mathbb{R}_+^{SG}, \mathbb{R}_+^{SM}) \times \mathbb{R}_{++}^{HN} \to \mathbb{R}^N$ defined by the formula

$$z^*(p, Q; a(p), e) = F^1(p; 1) - e^1 + \sum_{h=2}^{H} (F^h(p, Q; a(p), e^h) - e^h), \tag{6.2}$$

where F^h are individual demand functions. Then, $z^*(p,Q;a(p),e)=0$ if and only if the aggregate excess demand is zero and $pe^1=1$. Notice the function $\psi:\mathbb{R}^N_{++}\times O^{S,S-M}\to(\mathbb{R}^{S-M})^M$ defined above in the proof of Theorem 1 by $\psi(p,Q)=QV(p)$ is such that $\langle Q^{\mathrm{T}}\rangle^{\perp}\supset\langle V(p)\rangle$ is equivalent to $\psi(p,Q)=0$. Define the function $h:\mathbb{R}^N_{++}\times O^{S,S-M}\times\mathbb{R}^{HN}_{++}\times C^1(\mathbb{R}^{SG}_+,\mathbb{R}^{SM}_+)\to\mathbb{R}^N\times(\mathbb{R}^{S-M})^M$ by $h(p,Q;e,a(p))=(z^*(p,Q;a(p)\,e),\psi(p,Q))$. Define the space of economies, $A=C^1(\mathbb{R}^{SG}_+,\mathbb{R}^{SM}_+)\times\mathbb{R}^{HN}_{++}$. Define the set M of pseudo-equilibrium prices corresponding to an economy $E((u^h,e^h),a(p))$ with $(a(p),e)$ in A:

$$M(a(p),e)=\{(p,Q)\in\mathbb{R}^N_{++}\times O^{S,S-M}\,|\,h(p,Q;a(p),e)=0\}. \tag{6.3}$$

We define $X=\mathbb{R}^N_{++}\times O^{S,S-M}$, $Y=\mathbb{R}^N\times(\mathbb{R}^{S-M})^M$, and $\rho:A\to C^1(X,Y)$ by $\rho(a(p),e)(x)=h(a(p),e;x)$ with $x=(p,Q)\in X$. Note X is a C^1 manifold with boundary. A is an open set in the product of the Whitney C^1 and the Euclidean topologies. Hence A is a manifold, as is Y. Notice that X is of finite dimension, as is the codimension of the origin in Y. Notice X and A are both second countable. Define the evaluation map $\mathrm{ev}\rho:A\times X\to Y$ by $\mathrm{ev}\rho(a(p),e;x)=\rho(a(p),e)(x)$. It is obviously C^1, meaning that ρ is a C^1 representation. Define $\rho_{(a(p),e)}:X\to Y$ as follows, $\rho_{(a(p),e)}(x)=\mathrm{ev}\rho(a(p),e;x)$. Note that the set of pseudo-equilibrium prices, $M(a(p),e)=\rho^{-1}_{(a(p),e)}(0)$. By the openness of transversal intersection theorem [18.2 in Abraham and Robbin (1967)], the set A_0 that is defined by $A_0=\{(a(p),e)\in A\,|\,\rho_{(a(p),e)}$ is transverse to $0\}$ is open in A. In order to apply the transversal density theorem, it remains to be shown that $\mathrm{ev}\rho$ is transverse to 0. This is equivalent to showing 0 is a regular value of $\mathrm{ev}\rho$. The proof can now be broken up into five natural steps:

Step 1. $0\in Y$ is a regular value of $\mathrm{ev}\rho:A\times X\to Y$.

By definition, 0 is a regular value of $\mathrm{ev}\rho$ if $D\mathrm{ev}\rho:T_{(a(p),e)}A\times T_XX\to T_0Y$ is onto for all $(a(p),x)\in\mathrm{ev}\rho^{-1}(0)$. But, $D\mathrm{ev}\rho=(D_1\mathrm{ev}\rho,D_2\mathrm{ev}\rho)$ with subscripts denoting partial derivatives. So, $D_1\mathrm{ev}\rho(a(p),e,x):C^1(\mathbb{R}^{SG}_+,\mathbb{R}^{SM}_+)\times\mathbb{R}^{HN}\to\mathbb{R}^N\times(\mathbb{R}^{S-M})^M$ is defined as follows: $D_1(\mathrm{ev}\rho(a(p),e,x)(a^*(p),e^*)=D\mathrm{ev}\rho(a(p),e,x)(a^*(p),e^*,0)$ for $0\in T_{X^*}X$. Similarly, we define $D_2\mathrm{ev}\rho(a(p),e,x)$: $\mathbb{R}^N\times O^{S,S-M}\to\mathbb{R}^N\times(\mathbb{R}^{S-M})^M$. By definition, $\mathrm{ev}\rho((a(p),e);p,Q)=(z^*(p,Q;e),\psi(p,Q;a(p)))$. So we can define the coordinate functions $\mathrm{ev}\rho^1((a(p),e);p,Q)=z^*(p,Q;e)$ and $\mathrm{ev}\rho^2((a(p),e);p,Q)=\psi(p,Q;a(p))$. That 0 is a regular value of $\mathrm{ev}\rho$ means that for all $c\in\mathbb{R}^N$ and for all $d\in(\mathbb{R}^{S-M})^M$; $\exists a^*(p)\in C^1(\mathbb{R}^{SG}_+,\mathbb{R}^{SM}_+)$, $e^*\in\mathbb{R}^{HN}$, and $x^*\in\mathbb{R}^N\times O^{S,S-M}$ such that:

$$D_1\mathrm{ev}\rho^1(a(p),e,x)(a^*(p),e^*)+D_2\mathrm{ev}\rho^1(a(p),e,x)(x^*)=c, \tag{6.4}$$

$$D_1\mathrm{ev}\rho^2(a(p),e,x)(a^*(p),e^*)+D_2\mathrm{ev}\rho^2(a(p),e,x)(x^*)=d. \tag{6.5}$$

Notice that for all $(p,Q;e,a(p)),D_{a(p)}\mathrm{ev}\rho^1=\partial z^*/\partial a(p)=0$ and $\partial\mathrm{ev}\rho^1/\partial e^1=$

$\partial z^*/\partial e^1 = -I$, where I is an N by N square matrix, so that eq. (6.4) can be solved for (p^*, Q^*, e^*) given any $c \in \mathbb{R}^N$. As for eq. (6.5), notice that $D_1 \mathrm{ev}\rho^2(a(p), e, x)(a^*(p), e^*) = QV^*(p)$, where $V^*(p) = a^*(p(s))$. This means that eq. (6.5) can be rewritten as

$$QV^*(p) = d - D_2 \mathrm{ev}\rho^2(a(p), e, x)(x^*). \tag{6.6}$$

This is equivalent to

$$QV^*(p) = d - [D_p a(p^*(s))] - QV^*(p). \tag{6.7}$$

Once (p^*, Q^*, e^*) has been solved for in eq. (6.4), eq. (6.7) is of the form $QV^*(p) = k$ for some $k \in (\mathbb{R}^{S-M})^M$. But, $QV^*(p)$ is an onto map because $a^*(p)$ can be chosen so that $QV^*(p) = k$, for all $Q \in O^{S, S-M}$ and for all $k \in (\mathbb{R}^{S-M})^M$.

Step 2. A_0 is residual in A.

In order to apply the transversal density theorem [19.1 of Abraham and Robbin (1967)], we have to verify condition (3) of it [Abraham and Robbin, p. 48], namely $r > \max(0, \dim X - \dim Y)$; in our case, $r = 1$. The dimension of the range $Y = \mathbb{R}^N \times (\mathbb{R}^{S-M})^M$ is $N + (S-M)M$. As for the domain X, its dimension must be calculated in light of the fact that both the modified excess demand and security market functions are $O(S-M)$ invariant. We have already noted after eq. (3.1), the budget correspondences are $O(S-M)$ invariant. So are the demand functions constructed from the budget correspondences as well as the corresponding (modified) aggregate excess demand function. As for the security market function, by imposing the subspace $\langle Q^{\mathrm{T}} \rangle^{\perp}$ on the spot market budget constraints of individuals, the spot market equilibrium price p that results depends on Q, meaning $p = p(Q)$. Because this price only depends on the M-dimensional subspace and not on its representation, $\forall g \in O(S-M)$, $p(gQ) = p(Q)$. The asset market function can be rewritten to emphasize the dependence of p on Q: $\psi(p, Q) = QV(p(Q))$. Then as p only depends on the subspace of achievable income transfers and not its representation, this holds $\forall g \in O(S-M)$ and $\forall Q \in O^{S, S-M}$:

$$\psi(p, gQ) = gQV(p(gQ)) = gQV(p(Q)) = g\psi(p, Q). \tag{3.7}$$

Thus, $\forall x = (p, Q) \in \mathbb{R}^N_{++} \times O^{S, S-M}$, we have that h is $O(S-M)$ invariant; that is.

$$h(p, Q; e, a(p)) = h(p, gQ; e, a(p)) \;\; \forall g \in O(S-M). \tag{6.8}$$

This means that in calculating the dimension of X, the domain of h, we must subtract the dimension of $O(S-M)$, the orthogonal group of square $(S-M)$ matrices, because two matrices Q and Q' represent the same subspace $\langle Q^{\mathrm{T}} \rangle^{\perp}$ if for some $g \in O(S-M)$, $Q = gQ'$. Thus, the dimension of the domain is N

plus the dimension of the Stiefel manifold minus the dimension of $O(S-M)$. The dimension of the Stiefel manifold is given by dim $O^{S,S-M}=S(S-M)-(S-M)(S-M+1)/2$ [see, for example, Dubrovin et al. (1985, p. 44)], while the dimension of the orthogonal group is given by dim $O(S-M)=(S-M)(S-M-1)/2$ [for details, see Auslander and Mackenzie (1963, pp. 133–134)].

So, dim $X=[N+S(S-M)-(S-M)(S-M+1)/2]-\dim O(S-M)=[N+S(S-M)-(S-M)(S-M+1)/2-(S-M)(S-M-1)/2]=[N+(S-M)(S-S+M)]=[N+M(S-M)]$. But, dim $Y=[N+M(S-M)]$. So, dim $X-\dim Y=0$. Thus, A_0 is residual (and hence dense) in A.

Step 3. For all $(a(p),e)\in A_0, \rho^{-1}_{(a(p),e)}(0)$ is a submanifold (in X) of dimension zero.

By Corollary 17.2 of Abraham and Robbin (1967), for all $(a(p),e)\in A_0, \rho^{-1}_{(a(p),e)}(0)$ is a submanifold (in X) of the same codimension as $0\in Y$, or dim $\rho^{-1}_{(a(p),e)}(0)=\dim X-\dim Y=0$, and $M(a(p),e)$ is (a non-empty set by Theorem 1) of dimension zero.

Step 4. Generically, for all $j=1,\ldots,G; p^j(s)\neq p^j(s^*)$ for all $s\neq s^*$.

Note the negation of the above statement is that $\exists j\in\{1,\ldots,G\}$ and $\exists s,s^*$ such that $p^j(s)=p^j(s^*)$. If that were true, then we can use commodity j and states s and s^* to define a map $\Gamma: A\to C^1(X,R)$ as follows, $\Gamma(a(p),e)(p,Q)=(p^j(s)-p^j(s^*))$. Define another map $\eta: A\to C^1(X,Y\times R)$ by $\eta(a(p),e)(x)=(\rho(a(p),e), \Gamma(a(p),e))(x)$. Then, we define the evaluation map $\mathrm{ev}\eta: A\times X\to Y\times R$ by $\mathrm{ev}\eta(a(p),e;x)=\eta(a(p),e)(x)$. Note $\mathrm{ev}\eta=(\mathrm{ev}\rho,\mathrm{ev}\Gamma)$, where $\mathrm{ev}\Gamma$ is defined: $\mathrm{ev}\Gamma: A\times X\to R$ by $\mathrm{ev}\Gamma(a(p),e;x)=\Gamma(a(p),e)(x)$. We define $\eta_{(a(p),e)}: X\to Y\times R$ by $\eta_{(a(p),e)}(x)=\mathrm{ev}\eta(a(p),e,x)$. $\eta^{-1}_{(a(p),e)}(0)$ is that subset of $M(a(p),e)=\rho^{-1}_{(a(p),e)}(0)$ for which the first-period price is the same for two states of nature s and s^*, and for some commodity j. A natural question that arises is whether $0\in\mathbb{R}^N\times\mathbb{R}^{SM}\times\mathbb{R}$ is a regular value of $\mathrm{ev}\eta$, that is, whether $\mathrm{ev}\eta^{-1}(0)$ is a (sub)manifold. If we define v as $(a(p),e;x)$, this means asking if for all $v\in\mathrm{ev}\eta^{-1}(0)$, $D_v\mathrm{ev}\eta$ has maximal rank. But, from step 1, we know that for all $v\in\mathrm{ev}\rho^{-1}(0)$, the rank of $D_v\mathrm{ev}\rho=N+SM$. We will show that the rank of $D_v\mathrm{ev}\Gamma=1$. Together, these two facts imply that for all $v\in\mathrm{ev}\eta^{-1}(0)$, the rank $D_v\mathrm{ev}\eta=N+SM+1$, or equivalently, $D_v\mathrm{ev}\eta$ is onto. As $\mathrm{ev}\Gamma$ does not depend on $a(p)$, e or Q, we have $D_{a(p)}\mathrm{ev}\Gamma=D_e\mathrm{ev}\Gamma=D_Q\mathrm{ev}\Gamma=0$. Note $D_p\mathrm{ev}\Gamma$ is an N-dimensional vector having zeroes for all entries that are partial derivatives of $\mathrm{ev}\Gamma$ with respect to $p^k(s)$, for $k\neq j$ and for all s; 1 for the entry that is the partial derivative of $\mathrm{ev}\Gamma$ with respect to $p^j(s)$; and -1 for the entry that is the partial derivative of $\mathrm{ev}\Gamma$ with respect to $p^j(s^*)$; and zeroes for all the entries that are partial derivatives of $\mathrm{ev}\Gamma$ with respect to $p^j(s')$ if $s'\neq s$ or s^*. Thus the rank of $D_p\mathrm{ev}\Gamma$ is 1. So is the rank of $D_v\mathrm{ev}\Gamma$. Therefore, $0\in\mathbb{R}^N\times\mathbb{R}^{SM}\times\mathbb{R}$ is a regular value of $\mathrm{ev}\eta$.

Define $A_1 = \{(a(p), e) \in A \mid \eta_{(a(p),e)}$ is transverse to $0\}$. By the openness of transversal intersection theorem (18.2 in Abraham and Robin), A_1 is open in A. And as before with A_0, by the transversal density theorem (Theorem 19.1 of Abraham and Robbin) we know that A_1 is residual (and hence dense) in A. By Corollary 17.2 of Abraham and Robin, we know that for all $(a(p), e) \in A_1, \eta^{-1}_{(a(p),e)}(0)$ is a submanifold (in X). In fact, for all $(a(p), e) \in A_1$, we already know the dimension of $\eta^{-1}_{(a(p),e)}(0) =$ dimension $X -$ dimension $(Y \times R)$, which is -1, because we know from step 2 that dimension $X =$ dimension Y. Thus, for all $(a(p), e) \in A_1$, no pseudo-equilibrium of the economy $E((u^h, e^h, \gamma^h), a(p))$ satisfies the additional property of having the first period price of some commodity j being equal for two states of nature s and s^*.

Step 5. Let $\Omega = A_0 \cap A_1$. Because they are both open and dense in A, Ω is an open and dense set in A. By the above for all $(p, Q) \in M(a(p), e))$ satisfying $(a(p), e) \in \Omega$, $\langle Q^{\mathrm{T}} \rangle^{\perp} = \langle V(p) \rangle$. Therefore, if $(a(p), e) \in \Omega$, then every pseudo-equilibrium of the economy $E((u^h, e^h), a(p))$ is regular. Q.E.D.

Proof of Theorem 3. It is well-known that a competitive equilibrium can be constructed from a regular pseudo-equilibrium. [see Duffie and Shafer (1985) or Husseini et al. (1990) for details.] Thus, by Theorem 2, for $(a(p), e) \in \Omega$, the set of competitive equilbiria is nonempty. Q.E.D.

Proof of Theorem 4. First, we eliminate the security prices by noting that the period zero budget constraint (2.1) can be rewritten by using the no-arbitrate condition (2.5):

$$p_0[x_0^h - e_0^h] + \sum_{j=1}^{M} \left[\sum_{s=1}^{S} \beta_s a_{js}(p(s)) \right] (\theta_j^h) = 0.$$

Using the period one budget constraint (5.2), this can be rewritten

$$p_0[x_0^h - e_0^h] + \sum_{s=1}^{S} \beta_s \{p(s)[x^h(s) - w^h(s)]\} = 0.$$

If we define $w^h(0) = e^h(0)$, we can rewrite the above equation as

$$p_0[x_0^h - e_0^h] + \sum_{s=1}^{S} p(s)[x^h(s) - w^h(s)] = 0, \quad \text{or} \quad p(x^h - w^h) = 0.$$

Define the budget correspondence

$$B: \Delta_+^{N-1} \times O^{S,S-M} \times \mathbb{R}_{++}^N \to \mathbb{R}_+^N \quad \text{for} \quad h = 2, \ldots, H$$

as follows:

$$B(p, Q; w^h) = \{x \in \mathbb{R}_+^N \mid p(x^h - w^h) = 0 \text{ and } p(1) \square [x^h(1) - w^h(1)] \in \langle Q^{\mathrm{T}} \rangle^{\perp}\}. \tag{6.9}$$

Then, we can define a pseudo-equilibrium for the economy $E((u^h, e^h, \gamma_0^h), a(p))$ is a three-tuple $(x, p, Q) \in \mathbb{R}_+^{HN} \times \Delta_+^{N-1} \times O^{S, S-M}$, consisting of an allocation of goods, prices for goods, and a trial subspace of income transfers such that

$$x^1 = \arg\max u^1(x^1) \quad \text{subject to} \quad p(x^1 - w^1) = 0, \tag{6.10}$$

$$x^h = \arg\max\, u^h(x^h) \quad \text{subject to} \quad x^h \in B(p, Q; w^h) \quad \text{for} \quad h = 2, \dots, H; \tag{6.11}$$

$$\sum_h x^h = \sum_h e^h; \quad \text{and} \tag{2.2}$$

$$\langle Q^{\mathrm{T}} \rangle^{\perp} \supset \langle V(p) \rangle. \tag{3.4}$$

Theorem 1 guarantees that pseudo-equilibrium exists for any economy. As before, a pseudo-equilibrium (x, p, Q) is regular if $\langle Q^{\mathrm{T}} \rangle^{\perp} = \langle V(p) \rangle$. The only novelty in applying the proof of Theorem 2 here is the analogue of eq. (6.4) since $D_{a(p)} \mathrm{ev} \rho^1 = \partial z^* / \partial a(p)$ is no longer necessarily zero. But, $\partial \mathrm{ev} \rho^1 / \partial e^1 = \partial z^* / \partial e^1 = -I$, where I is an N by N square matrix, so the corresponding version of eq. (6.4) can be solved for $(a^*(p), p^*, Q^*, e^*)$ given any $c \in \mathbb{R}^N$ as before. The rest of the proof of Theorems 2 and 3 go through unchanged. Q.E.D.

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