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Notes, Comments, and Letters to the Editor

Upper Semi-Continuity of the Separating Equilibrium Correspondence

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We formulate separating equilibria with (many) signals as the set of zeroes of a suitable mapping and show continuity of this mapping for a suitable topology. We prove upper semi-continuity (usc) of the correspondence mapping the set of multidimensional signalling environments into their sets of separating equilibria. *Journal of Economic Literature* Classification Numbers: 022, 026. © 1989 Academic Press, Inc.

1. INTRODUCTION

Spence [11, 12] introduced models in which buyers use signals to infer product quality. Riley [8] then provided a general model of informational equilibria. Lately, there has been a resurgence of interest concerning such models of markets with asymmetric information from a game-theoretic viewpoint. Riley [9] exhibited sufficient conditions for the Pareto-dominant, zero-profit, and separating market signalling equilibrium to be the unique Nash equilibrium of a many principal, many agent game. The principals are the uninformed buyers who move first in announcing price schedule offers. The informed sellers are the agents who move second by responding via optimal signal choices.

Engers and Fernandez [2] show existence, uniqueness, and informational consistency of a generalization of Riley's [8] reactive equilibrium, proving, moreover, that outcome is the Pareto-dominant, zero-profit separating equilibrium. They also provide a game-theoretic interpretation for their generalized reactive equilibrium by specifying an extensive-form game having that outcome among its perfect Nash equilibria. Unfortunately they point out an infinity of perfect Nash equilibria exist for their game since no last mover exists.

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Both of the above games are termed games of screening by Stiglitz and Weiss [13] as uninformed buyers move first. A different scenario has informed sellers move first; these are called signalling games by Stiglitz and Weiss [13]. Cho and Kreps [1] analyze such a labor market signalling game having a continuum of signal choices, but only a pair of seller types. They introduce an intuitive criterion and prove the Pareto-dominant, zero-profit, and separating equilibrium is the unique outcome satisfying this criterion from among the infinite number of sequential equilibria of their game tree.

Most recently, Engers [3] has provided a generalization of the above signalling environments to multidimensional quality types and signal choices. Although the literature on more than just a single-dimensional signal is just beginning, examples include both Milgrom and Roberts [6] and Wilson [14] who study the use of price and not directly informative adverstising by a monopolist producer of a new good to signal product quality. Both of those works, however, still involve only a unidimensional quality attribute. Quinzii and Rochet [7] as well as Kohlleppel [4] show that for a multidimensional quality attribute space certain results in Riley [8] and Spence [11, 12] do not generalize (like the sufficient conditions required for existence).

In light of all these recent attempts providing a game-theoretic foundation supporting the concept of market signalling and information transfer in markets, this paper assumes the existence of Nash equilibrium in such markets. Instead of a particular extensive game form, we ask what conditions are required for the usc of the separating equilibrium correspondence? The answer is the only additional condition required besides those already ensuring equilibria exist is continuous differentiability of the signalling cost as a function of the signalling variables, not just differentiability. We prove the function taking elements of the space of multidimensional signalling environments into their Pareto-dominating separating signalling equilibrium is continuous. This is established for the multidimensional signalling model of Engers [3] that generalizes both Quinzii and Rochet's [7] multidimensional signalling model as well as Riley's [8] model of single-dimensional informational equilibrium.

Continuity of the function from the space of multidimensional signalling environments defined in Engers into the space of Pareto-dominant separating sets is proven. This follows directly from the closed graph property of the correspondence mapping such environments into their set of (not necessarily Pareto-dominant) separating equilibria. That in turn is a consequence of continuity of a function defined so that its zeroes are precisely separating sets. Such a function is analogous to a traditional excess demand function of general equilibrium theory for prices in a finite-dimensional simplex. There is a difference now because the objects being equilibrated are themselves infinite-dimensional functions. Nonetheless, a generalization of the standard method of proof applies. The implications of the closed graph property are the same as those usually given for the closed graph property of the Walrasian equilibrium correspondence.

2. The Space of Economies with Many Signals

Engers [3] considers a class of multidimensional signalling environments that form a superset of Riley's [8] general model of unidimensional informational equilibria, which in turn is a superset of Spencian [11, 12] market signalling models, Leland and Pyle's [5] analysis of informational asymmetries in financial markets, as well as Rothschild and Stiglitz's [10] study of competitive insurance markets. Also as mentioned already, Engers' [3] model is less restrictive than Quinzii and Rochet's [7] multidimensional signalling model as the latter is a special case of the former. For the sake of completeness, assumptions in Engers' [3] canonical model are listed below with only some minor notational changes. Let Q be the set of all types and Y be the set of all feasible signals, both possibly multidimensional spaces.

Assumption 1. Y is a compact metric space.

Assumption 2. The value function of buyers $V: Q \times Y \rightarrow R$ is bounded, and $\forall q$ in Q, $V(q, y)$ is continuous on Y. We define $\mathbf{p} = \inf\{V(q, y) | q$ in Q and y in $Y\}$, $p^s = \sup\{V(q, y) | q$ in Q and y in $Y\}$, and $P = [\mathbf{p}, p^s]$.

Assumption 3. The preferences of sellers are given by the utility function $U: Q \times Y \times P \rightarrow R$, where $\forall q$ in Q, $U(q, y, p)$ is continuous on $Y \times P$, and strictly increasing in p.

Assumption M1. *The set Q of types is a compact metric space.*

Assumption M2. *Y is a compact convex subset of R^n.*

Assumption M3. *$V: Q \times Y \rightarrow R$ is continuous and nondecreasing in each y.*

Assumption M4. *The preferences of sellers are represented by the utility function $U(q, y, p)$ defined to be $p - C(q, y)$, where C, the cost of signalling function, is differentiable with respect to y with the derivative being denoted by $MC(q, y)$. Both $C(q, y)$ and $MC(q, y)$ are continuous on $Q \times Y$.*

ASSUMPTION M5. *If p belongs to P and y belongs to Y, then if x is a convex combination of elements of the set $\{MC(q, y) | q$ in Q and $V(q, y) \leqslant p\}$ and $MC(q^*, y) \geqslant x$, then $V(q^*, y) \leqslant p$.*

ASSUMPTION M6. *S contains (y, p^s)* $\forall y$ on the boundary of Y.

Engers [3] provides a justification and interpretation of all these assumptions, in particular, M5 and M6. He also defines the following terms and one more assumption. An offer set refers to a closed subset of $Y \times P$. The game being envisioned is this: buyers who are initially uninformed first make offers, which are defined to be signal–price pairs (y, p) representing commitments by the buyers to pay p for a unit of the commodity in question to each seller with the signal choice y. Then in response, sellers choose among the offers the one that maximizes their utility $U(q, y, p)$. As an offer set is a closed subset of a compact metric space, it also is compact and hence faced with a nonempty offer set A, there is a utility-maximizing choice for each type because U is continuous. To deal with possible non-uniqueness Engers [3] uses the following tie-breaking rule: type q sellers faced with more than a single optimal choice from an offer set A will be assumed to break ties by (i) picking that seller utility maximizing offer that also maximizes buyer profit, $V(q, y) - p$; and (ii) if a tie still remains, choose the lowest remaining choice according to a given ordering defined on $Y \times P$. This rule defines a function denoted by $c(q, A)$ or $t_A(q) = (y_A(q), p_A(q))$ for any nonempty offer set A and type q in Q because the set of those offers maximizing U, a continuous function over a compact set, is both itself compact and nonempty (by the Weierstrass theorem).

Let us follow Engers [3] and denote the profit buyers make on a seller of type q as $\pi_A(q)$ or $V(q, y_A(q)) - p_A(q)$. Following him, the convention that $\pi_A(q) = 0$ for all q if $A = \varnothing$ is adopted. If A is a nonempty offer set with $\pi_A(q) = 0$ for all q in Q, then we say A is a separating set. In this case, $V(q, y_A(q)) = p_A(q)\ \forall q$ in Q, so that sellers of different qualities choose different offers, and hence different signal levels, or, $y_A(q)$ as U is an increasing function of p. B is called a sure thing if B is an offer set and $\pi_B(q) \geqslant 0\ \forall q$ in Q. The sure offer set (denoted by S) is the union of all sure things. Engers [3] proves that S, the sure offer set, is not empty, itself a sure thing, and contains every separating set under Assumptions 1–3. Finally, he defines:

DEFINITION. B is a sure thing given A if A is an offer set, B is a nonempty subset of $Y \times P$, disjoint from A, with $A \cup B$ an offer set, and $\pi_{A \cup B}(q) \geqslant 0$ $\forall q$ in Q such that $c(a, A \cup B)$ belongs to B.

ASSUMPTION 4. *For any offer set A satisfying $\pi_A(q) > 0$ for some q in Q, there is a sure thing given A.*

Engers proves that either Assumptions 1–4 or Assumptions 1–3 and M1–M6 imply the sure offer set S is a separating set and contains all separating sets, hence Pareto-dominating them. While requiring fewer assumptions and therefore more aesthetically pleasing ones, using Assumptions 1–4 to prove Engers' main theorem also suffers from the drawback that Assumption 4 is not phrased in terms of the basic exogenous data like Assumptions 1–3 and M1–M6 are. The M5 condition is a quasiconvexity assumption that generalizes the unidimensional assumption of monotonicity of the marginal cost function. Engers [3] also provides a counterexample that S the sure offer set is the Pareto-dominant separating set if M5 fails to hold. Engers and Fernandez [2] do likewise when M6 is not satisfied.

We define now the space of multidimensional signalling environments to be the set M of all ordered pairs (U, V) satisfying the above listed Assumptions 1–3 and M1–M6. A separating equilibrium is a pair of functions $t_A(q) = (y_A(q), p_A(q))$, where A is a separating set for which this pair of functional equations hold simultaneously $\forall q$ in Q:

$$y_A(q) = \arg\max\{U(q, y_A(q), p_A(q)) | q \text{ in } Q\} \tag{2.1}$$

$$p_A(q) = V(q; y_A(q)). \tag{2.2}$$

Let $X = C(Q, Y) \times C(Q, R)$. For a given in M, $X \supset S(m)$ which denotes the set of separating equilibria corresponding to a particular multidimensional signalling environment, m. It follows from Engers [3, Theorem 1 or 2] that $\forall m$ in M, $S(m) \neq \varnothing$ and that $S(m)$ belongs to X. Define $\Psi_m: X \to C(Q, R^2)$, the set of continuous maps from Q into R^2:

$$\Psi_m(y, p)(q) \equiv [D_y\{U(q, y_A(q), p_A(q)\}, p_A(q) - V(q; y_A(q))]. \tag{2.3}$$

Here D_y stands for the Jacobian matrix of partial derivatives of U with respect to the y. Engers [3] showed that $\Psi_m^{-1}(0) = S(m)$, where 0 stands for the function on Q identically equal to the origin in R^2. That our necessary first-order conditions are, in addition, sufficient follows because Assumption A5 means C is quasiconvex or, equivalently, the objective function U of the informed sellers is quasiconcave (strict quasiconvexity of C or equivalently strict quasiconcavity of U would imply a unique maximum). Study of informational equilibria reduces to understanding properties of this map Ψ_m, such as continuity in a natural metric. Endow X with the Whitney C^1 uniform convergence norm and its associated metric and induced topology in order to have a sense of closeness for separating equilibria (candidates). Likewise, let M inherit the Whitney C^1 uniform norm, metric, and topology from its natural ambient space, namely that of all pairs of possible utility functions (U, V) with none of the Assumptions 1–3 and M1–M6 necessarily met.

3. Upper Semi-Continuity of the Separating Equilibrium Correspondence

Define the map $\Psi\colon M \times X \to C^1(Q, R^2)$ by $\Psi(m, x) \equiv \Psi_m(x)$. Then we have:

Lemma. *$\Psi\colon M \times X \to C^1(Q, R^2)$ is C^0.*

Proof. Consider arbitrary functions $n = (A, B)$ in M and $z = (g, h)$ in X. By definition, $\Psi(n, z) = \Psi^1(n, z), \Psi^2(n, z))$, where

$$\Psi^1(n, z)(q) = D_h\{A(q, h(q), g(q))\}, \tag{3.1}$$

$$\Psi^2(n, z)(q) = g(h(q)) - B(q, h(q)). \tag{3.2}$$

Taking the limits of both expressions above when $n \to m$ and $z \to x$ results in the limit $\Psi(e, x) = (\Psi^1(e, x), \Psi^2(e, x))$, since $A \to U$, $B \to V$, $g \to p$, and $h \to y$ as functions in the Whitney C^1 metric means their values get close as well as those of their partial derivatives taken with respect to the signals in the case of the partials of A with respect to h. This property implies the closed graph property of the separating equilibrium correspondence:

Theorem. *The correspondence $S\colon M \to X$ is usc.*

Proof. Let m belong to M, $\{m_n\}$ in M be a sequence such that $m_n \to m$, $\{x_n\}$ in X be a sequence such that for all n, x_n belongs to $S(m_n)$, and let $x_n \to x$ in X. It must be shown that x belongs to $S(m) \Leftrightarrow \Psi(m, x) = 0$ to prove the usc of S. But the fact that Ψ is C^0 means that: $\Psi(m, x) = \Psi(\lim m_n, \lim x_n) = \lim \Psi(m_n, x_n) = 0$.

We note that our above result can be specialized to show the function mapping a multidimensional signalling environment into its unique Pareto-undominated informational equilibrium is continuous. This is because all of these Pareto-undominated separating equilibria satisfy the same boundary or initial condition.

Corollary. *$\forall \varepsilon > 0, \exists \delta > 0$ such that $d(m_1, m_2) < \delta \Rightarrow \eta(\mathrm{SPD}(m_1), \mathrm{SPD}(m_2)) < \varepsilon$, where $\mathrm{SPD}(m)$ is the Pareto-dominant separating equilibrium that corresponds to multidimensional signalling environment m and η is the Hausdorff metric induced by the Whitney C^1 metric on X.*

Proof. This is an immediate consequence of the above theorem.

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