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THE ROBUSTNESS OF MULTIDIMENSIONAL SIGNALLING EQUILIBRIA *

Peter H. HUANG

Tulane University, New Orleans, LA 70018, USA

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This paper demonstrates upper hemi-continuity of a multidimensional signalling equilibrium correspondence.

1. Introduction

The question of robustness or structural stability is an important one to ask of any equilibrium concept because of the desirability of continuity (and even differentiability, if possible) in order to do sensitivity analysis (and qualitative if not quantitative comparative statics). Such a question has been resolved affirmatively for Riley's (1979) one-dimensional concept of informational equilibrium by Huang (1984). The importance of this result for the recent dividend signalling models based on Riley (1979) is explained in Huang (1987). In this paper such a finding is extended to a framework of multidimensional informational asymmetries. There have recently been a number of such models. These include McAfee and McMillan's (1987) study of incentive compatibility and mechanism design in general for multidimensional quality types. Kohlleppel (1983a, b) provides an example of the non-existence of signalling equilibrium in the Spence (1974) model with both the dimension of possible types and signals equal to two. Ambarish, John and Williams (1987) as well as Hughes (1986) follow the tradition of Leland and Pyle (1977) in the context of providing financial signalling models. Finally, within the field of industrial organization Baron and Myerson (1982) as well as more recently Milgrom and Roberts (1986) in addition to Wilson (1985) provide multidimensional signalling models.

To be concrete in this paper we shall follow the particular notation and structure of the model in Quinzii and Rochet (1985): $n = (n_1, \dots, n_k) \in N$ with $N =$ set of all possible hidden k-dimensional quality types of individuals; while $y = (y_1, \dots, y_k) \in R^k_+$ represents the possible observed investment levels by workers in k (educational) signals. There is a single constant returns to scale production technology which is available to all firms, namely $s(n, y)$. In addition, we assume that up to a change of variables the costs of signalling are linear and separable in y, that is $c(n, y) = \Sigma_i(y_i/n_i)$. Every member of a continuum of workers is assumed to be choosing y to maximize the net income from signalling. Now introduce the variables $x \in \Omega$, an open convex subset of R^k_+(actually, we assume that $\Omega = R^k_{++}$): $x_i = 1/n_i$, for $i = 1, \dots, k$; this means individuals maximize $H(x, y)$ where their net income $H(x, y)$ is $w(y) - \Sigma_i(x_i y_i)$.

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Now define a (separating) signalling equilibrium to be a pair of functions, (y, w), where y: $N \to R_+^k$ and w: $R_+^k \to R_+$ such that $\forall\ n \in N$, these conditions are satisfied:

(a) $y(n) = \arg\max\ \{w(y) - c(n, y(n))\}$ over R_+^k, i.e., individual rationality,
(b) $w(y(n)) = s(n, y(n))$, i.e., rational expectations of employer beliefs.

Quinzii and Rochet (1985) prove the following three results concerning the characterization, existence, and game-theoretic 'stability' of such separating signalling equilibria:

Theorem 1. There is a one-to-one correspondence between signalling equilibria and convex non-negative non-increasing solutions of the partial differential equation: $u(x) = H(x, -Du(x))$ *for a.e.* $x \in \Omega$.

Theorem 2. Assume the following four conditions are satisfied:

A.1. s is continuously differentiable.

A.2. $\forall\ (x, y)$ *and* $\forall\ i = 1,\dots, k$, $(\partial s/\partial x_i)(x, y) \le 0$ *and* $\exists$ *a positive constant, M such that* $0 \le (\partial s/\partial y_i)(x, y) \le M$.

A.3. $\forall\ x \in \Omega$, $y \to H(x, y)$ *is concave and has a maximum on* R_+^k.

A.4. $\forall\ y \in R_+^k$, $x \to H(x, y)$ *is convex.*

Then, there exists a (separating) signalling equilibrium.

Theorem 3. Under assumption A.4 above, at a pooling, zero profit equilibrium, there is an alternative offer of a wage schedule yielding positive expected profits.

2. The space of multidimensional signalling markets

We define S to be the set of all functions s: $N \times R_+^k \to R_+$ satisfying assumptions A.1–A.4 and some further mild technical conditions. [1] Thus, if $\forall\ s \in S$, we denote the set of signalling equilibrium by $E(s)$, Theorem 2 tells us that $\forall\ s \in S$, $E(s) \neq \varnothing$. The range of E is the set Y of pairs of functions (y, w) with y: $N \to R_+^k$ and w: $R_+^k \to R_+$.

A natural question is to ask whether our equilibrium correspondence E: $S \to Y$ is upper hemi-continuous [2] (u.h.c.) with respect to some appropriate choice of topologies on S and Y. The answer is yes, if we use the compact-open C^0 topology on S and Y. By Theorem 1 we associate $E(s)$ with the solutions to $u(x) = H(x, -Du(x))$ for a.e. $x \in \Omega$, with $u(x) \in P$, where P is the set of differentiable real-valued functions on Ω. Defining $F_s(u) = u(x) - H(x, -Du(x))$, where s enters in the definition of H, we can identify $E(s)$ with the zeroes of the map $F_s(u)$ over U, which is a compact set in the C^1 topology containing all of the convex, non-negative, and non-increasing real-valued functions over Ω. Hence the study of the set of signalling equilibria $E(s)$ reduces to studying the solutions to $F_s(u) = 0$ for a.e. $x \in \Omega$. We now define another map $F(s, u) = F_s(u)$ so $F(s, u) = 0 \Leftrightarrow y(x) = -Du(x)$ and

[1] In order to ensure that S is open in A.2 we assume all the inequalities on the partials of s hold strictly [that $\exists K > 0$ s.t. $(\partial s/\partial y_i)(x, y) < K \forall\ (x, y)$ is no problem – just choose $K > M$; while the other two positivity of s_x and s_y conditions do actually restrict our work and exclude the so-called pure signalling case where $s_y = 0$]. We assume in addition $H(x, y)$ is strictly concave in y and strictly convex in x as well as s being C^2 in order to be able to express assumptions A.3 and A.4 in terms of restictions on the sign of particular Hessians of s.

[2] See Berge (1963) or Moore (1968) for a precise definition and complete discussion of the various notions of upper hemi-continuity appearing in the literature.

$w(y(x)) = u(x) - \Sigma_i(x_i\, \partial u_i(x)/\, \partial x_i)$ define a.e. $x \in \Omega$ a (separating) signalling equilibrium for the market described by the function pair of $s\&c(x, y) = \Sigma_i(x_i y_i)$. We are then able to prove:

Theorem A. $F: S \times U \to P$ *is continuous using the compact-open* C^1 *topology on* U *and the compact-open* C^0 *topology on* S *and* P.

Proof. As $F(s + at,\ u + av)(x) = u(x) + av(x) - s(x,\ -Du(x) - aDv(x)) - at(x,\ -Du(x) - aDv(x)) - \Sigma_i(x_i\, \partial u_i(x)/\, \partial x_i) - a\Sigma_i(x_i\, \partial v_i(x)/\, \partial x_i)$, the lim of $F(s + at,\ u + av)$ as $a \to 0$ is just $F(s,\ u)$ where $t \in S$ and $v \in U$. □

Theorem B. The correspondence from S *to* U *mapping an* s *into those* u *that are zeroes of* F_s *has a closed graph.* [3]

Proof. By Berge (1963, p. 111) or Moore (1968, p. 130) it suffices to prove that correspondence is closed in S, meaning that for each s in S, the hypothesis that $s_n \to s$ for a sequence $\{s_n\}$ in S such that $F(s_n,\ u) = 0\ \forall\ n$ implies the conclusion that $F(s,\ u) = 0$. But, upon direct verification $F(s,\ u) = F(\lim s_n,\ u) = \lim F(s_n,\ u) = \lim 0 = 0$. □

Theorem C. $E: S \to Y$ *is u.h.c. using the compact-open* C^0 *topology on* S *and* Y *(in addition to the compact-open* C^1 *topology on* U*).*

Proof. E is the composition of the correspondence in Theorem B and a continuous function. Berge (1963, p. 113) has proven the composition of two u.h.c. correspondences is also u.h.c. The above defined correspondence in Theorem B has already been shown to have a closed graph. The range of that correspondence, namely U, has been chosen w.l.o.g. to be compact so that would be a u.h.c. correspondence. The function from U to Y mapping each zero u in U of F_s to a pair $(y(x),\ w(y(x))$ that is a signalling equilibrium for s is defined in Quinzii and Rochet (1985): $y(x) = -Du(x)$ and $w(y(x)) = u(x) - \Sigma_i(x_i\, \partial u_i(x)/\, \partial x_i)$ a.e. $x \in \Omega$. Both of these formulae define continuous functions in u for the compact-open C^1 topology on U as can be seen directly: if $y(x) = b(u)(x) \doteq -Du(x)$, then lim of $b(u + av)(x) = \lim\ [-Du(x) - aDv(x)] = b(u)(x)$ when $a \to 0$. If $w(y(x)) = d(u)(x)$, then lim of $d(u + av)(x) = \lim\ [u(x) - \Sigma_i(x_i\, \partial u_i(x)/\, \partial x_i) + av(x) - a\Sigma_i(x_i\, \partial v_i(x)/\, \partial x_i) = d(u)(x)$ as $a \to 0$. □

3. Conclusions

In summary, then upper hemi-continuity of a multidimensional signalling equilibrium correspondence under perturbations of the s function has been shown in the case of no wealth effects, that is, a multidimensional signalling model à la Spence (1974) as opposed to Riley's (1979) more generalized preferences. Thus, although there are qualitative differences between models of asymmetric information with unidimensional versus multidimensional initial quality type of endowments, like the conditions required for existence of signalling equilibria, given those respective conditions, upper hemi-continuity of the signalling equilibria holds.

[3] This is based on a familiar argument for Walrasian general equilibrium, included here only for the sake of completeness.

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